Please read the Addendum at the end of this study unit before you start Lesson 1.
Welcome to *Mathematics Grade 12: Numbers and Number Relationships*. We have designed the lessons in this unit in such a way that each builds on the knowledge gained from the previous lessons. Therefore, it’s essential that you ensure that you have a good understanding of each lesson. It’s also important that you work through the lessons in the order in which we present them in the unit.

**The icons used in this study unit**

Read the descriptions of icons below. Look out for these icons as you work through the study unit. They will show you at a glance where you need to work through activities, definitions, self-assessment questions, and so on.

<table>
<thead>
<tr>
<th>Icon</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image" alt="Learning Outcomes" /></td>
<td><strong>Learning Outcomes:</strong> This signals the Learning Outcomes of the unit. You must be able to show competence in the Outcomes after you have worked through the study unit. Competence means that you must be able to demonstrate that you can meet an Outcome with skill and knowledge.</td>
</tr>
<tr>
<td><img src="image" alt="Definition" /></td>
<td><strong>Definition:</strong> This signals an important definition that you should understand and remember.</td>
</tr>
<tr>
<td><img src="image" alt="Mathematical formula" /></td>
<td><strong>Mathematical formula:</strong> This is a formula that you will use in calculations. It is important that you know what values to substitute into the formula.</td>
</tr>
<tr>
<td><img src="image" alt="Important statement" /></td>
<td><strong>Important statement:</strong> This signals an important point that you must grasp before you continue with the rest of the lesson. It could also signal an interesting snippet of information.</td>
</tr>
<tr>
<td><img src="image" alt="Activity" /></td>
<td><strong>Activity:</strong> This signals an activity that you should complete as part of your learning. We have interspersed activities throughout your lessons.</td>
</tr>
<tr>
<td><img src="image" alt="Self-assessment questions" /></td>
<td><strong>Self-assessment questions:</strong> This signals the questions that will help you to analyse your understanding of the theory that was covered in the lesson. We conclude each lesson with a set of self-assessment questions.</td>
</tr>
<tr>
<td><img src="image" alt="Answers to self-assessment questions" /></td>
<td><strong>Answers to self-assessment questions:</strong> This signals the suggested answers to the self-assessment questions. Please do not look at the answers before you have tried to answer the questions yourself.</td>
</tr>
<tr>
<td><img src="image" alt="Competence checklist" /></td>
<td><strong>Competence checklist:</strong> This signals a checklist to help you discover whether you can meet all the Assessment Standards in the lesson.</td>
</tr>
</tbody>
</table>
The best way to study

To ensure that you get the full benefit of this study unit, we recommend that you do the following:

• Carefully read the next section. It provides you with the Learning Outcomes and Assessment Standards for each of the lessons in the study unit.
• Carefully and diligently work through each lesson, keeping in mind the Outcomes that you have to achieve.
• Ensure that you complete all the activities in the lessons.
• Ensure that you answer all the self-assessment questions in the lessons. Compare your answers to the answers that we provide in this study unit.
• If you encounter any words that you do not understand, make a list of them, and then either look them up in a dictionary or ask your tutor for their meanings.
**Learning Outcomes and Assessment Standards**

*Numbers and Number Relationships* has been developed according to the National Curriculum Statement subject guidelines for Mathematics Grade 12. We also included the National Curriculum Statement subject guidelines for Mathematics Grade 11 as revision as this will also be assessed in the final examination. Read through the lists below. Your aim is to complete the Learning Outcome and Assessment Standards for this unit successfully. In this unit, you will complete one Learning Outcome and six Assessment Standards for Grade 11 (Revision) and five Assessment Standards for Grade 12. You need to have this information because you must keep checking your learning goals.

<table>
<thead>
<tr>
<th>Learning Outcomes and Assessment Standards</th>
</tr>
</thead>
<tbody>
<tr>
<td>After you have worked through the content covered in this study unit, you should be able to do all the tasks we have listed below:</td>
</tr>
<tr>
<td>Learning Outcome 1: <strong>Numbers and Number Relationships</strong></td>
</tr>
<tr>
<td>When solving problems, the learner is able to recognise, describe, represent and work confidently with numbers and their relationships to estimate, calculate and check solutions.</td>
</tr>
<tr>
<td><strong>Assessment Standards (Grade 11 Revision)</strong></td>
</tr>
<tr>
<td>11.1.1 Understand that not all numbers are real. (This requires the recognition, but not the study of non-real numbers)</td>
</tr>
<tr>
<td>11.1.2 (a) Simplify expressions using the laws of exponents for rational exponents.</td>
</tr>
<tr>
<td>(b) Add, subtract, multiply and divide simple surds.</td>
</tr>
<tr>
<td>(c) Demonstrate an understanding of error margins.</td>
</tr>
<tr>
<td>11.1.3 Investigate number patterns (including, but not limited to those where there is a constant second difference between consecutive terms in a number pattern, and the general term is therefore quadratic) and hence:</td>
</tr>
<tr>
<td>(a) make conjectures and generalisations</td>
</tr>
<tr>
<td>(b) provide explanations and justifications, and attempt to prove conjectures.</td>
</tr>
<tr>
<td>11.1.4 Use simple and compound decay formulae ( A = P(1-ni) ) and ( A = P(1-i)^n ) to solve problems (including, straight line depreciation and depreciation on a reducing balance).</td>
</tr>
<tr>
<td>11.1.5 Demonstrate an understanding of different periods of compounding growth and decay (including, effective compounding growth and decay and including effective and nominal interest rates).</td>
</tr>
<tr>
<td>11.1.6 Solve non-routine, unseen problems.</td>
</tr>
</tbody>
</table>
Assessment Standards (Grade 12)

12.1.2 Demonstrate an understanding of the definition of a logarithm and any laws needed to solve real-life problems (for example, growth and decay).

12.1.3 (a) Identify and solve problems involving number patterns (including, but not limited to) arithmetic and geometric sequences and series.
(b) Correctly interpret sigma notation.
(c) Prove and correctly select the formula for, and calculate the sum of series, including:

\[ \sum_{i=1}^{n} 1 = n \]

\[ \sum_{i=1}^{n} i = \frac{n(n+1)}{2} \]

\[ \sum_{i=1}^{n} a + (i-1)d = \frac{n}{2} [2a + (n-1)d] \]

\[ \sum_{i=1}^{n} a \times r^{i-1} = \frac{a(r^n - 1)}{r - 1}; r \neq 1 \]

\[ \sum_{i=1}^{\infty} a \times r^{i-1} = \frac{a}{1-r}; -1 < r < 1 \]

(d) Correctly interpret recursive formulae (for example, \( T_{n+1} = T_n + T_{n-1} \)).

12.1.4 (a) Calculate the value of \( n \) in the formula \( A = P(1+i)^n \).
(b) Apply knowledge of geometric series to solving annuity, bond repayment and sinking fund problems, with or without the use of the formulae:

\[ F = \frac{x((1+i)^n - 1)}{i} \quad \text{and} \quad P = \frac{x(1 - (1+i)^{-n})}{i} \]

12.1.5 Critically analyse investment and loan options, and make informed decisions as to the best option(s), including, pyramid and micro-lenders’ schemes.

12.1.6 Solve non-routine, unseen problems.
INTRODUCTION TO THIS STUDY UNIT

In this study unit, *Mathematics Grade 12: Numbers and Number Relationships*, we'll cover Learning Outcome 1, as outlined in the National Curriculum Statement for Mathematics. Learning Outcome 1 reads as follows:

*When solving problems, the learner is able to recognise, describe, represent and work confidently with numbers and their relationships to estimate, calculate and check solutions.*

We will break the Learning Outcome Statement in this unit into different lessons. Some of the lessons include revision of Grade 11 work, and they will be highlighted as such.

In this study unit, you'll learn about numbers, and the relationships between numbers.

The first lesson is a revision of Grade 11 topics, and deals with indices or exponents. We will revise the laws of indices. We will also revise surds.

Then we'll introduce you to the concept of a logarithm. Here, you'll learn how to apply the laws of indices to logarithms. We will discuss the relationship between indices and logarithms, and explain when it is necessary to use logarithms.

The second task is for you to apply what you have learnt about numbers and number relationships to problems.

You'll learn how to estimate and calculate answers to problems, both with and without a calculator. Specifically, we'll look at problems related to business transactions like bonds, hire purchase and annuities. This will help you to develop the ability to explore real-life mathematical problems.

A basic concern for anyone responsible for managing money is to determine the future value of current investments. The only reason that you invest money is so that it will be worth more in the future.

The *time value of money* refers to the guarantee that R1 today is worth more than R1 at some time in the future.

In this unit, you will learn how and why money is worth more in the future by discussing the effect of interest. We will explore simple and compound interest, which is revision of Grade 11 work, in order to make sense of:

- effective and nominal interest rates; and
- appreciation and depreciation.

*Interest* refers to a charge made for the use of someone else's money. When we borrow money from other people or institutions, then they charge us interest. However, when we invest money with a financial institution, then they pay us interest. Banks usually charge interest as a percentage of the amount originally borrowed or invested.

You should read your guide before you start working through this study unit. Your guide tells you when and how to work through this study unit.

Your guide refers to this study unit in the study schedule, and at the start of the relevant study session. *This study unit is your prescribed learning material.*
To ensure the best learning experience, it is very important that you do all the activities and answer all the self-assessment questions that appear in this study unit.

To help you remember and to ensure that you have read all initial explanations and definitions it is a good idea to highlight the key words or formulas – especially in long explanations.
Learning Outcomes for Lesson 1

After you have worked through Lesson 1, you should be able to do the following:

<table>
<thead>
<tr>
<th>LO</th>
<th>Assessment Standard</th>
</tr>
</thead>
<tbody>
<tr>
<td>LO 1</td>
<td><strong>AS 11.1.1</strong> Understand that not all numbers are real. (This requires the recognition, but not the study of non-real numbers)</td>
</tr>
<tr>
<td></td>
<td><strong>AS 11.1.2</strong> (a) Simplify expressions using the laws of exponents for rational exponents.</td>
</tr>
<tr>
<td></td>
<td><strong>AS 11.1.6</strong> Solve non-routine, unseen problems.</td>
</tr>
<tr>
<td></td>
<td><strong>AS 12.1.6</strong> Solve non-routine, unseen problems.</td>
</tr>
</tbody>
</table>

We would like to help you to remain in complete control of your studies. So we give you an opportunity to check your competence at the end of each lesson. We've provided you with a checkbox to tick at the end of the lesson. When you tick the Assessment Standards, you'll see what you've achieved in each lesson!

**Introduction**

Lesson 1 is all about indices. After completing this lesson, you will:

- know the four basic definitions of indices;
- know the four laws of indices;
- be able to apply the definitions and laws without using a calculator; and
- be able to simplify expressions containing indices.

If you multiply a number, say 2, twelve times, we write it as follows:

\[ 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2. \]

As you can see, it takes up a lot of space to write the expression. Mathematicians have found an easier and shorter way to write this expression, by using **indices**.

**Indices** are an example of how mathematicians have found a clever way of expressing complex expressions in a relatively simple way. This index notation enables you to simplify mathematical expressions, and to solve complicated equations.

The mathematical expressions that we'll work with in this lesson will consist of either one term or will be a **polynomial**.

A **polynomial** is a mathematical expression that has more than one term.
You’ll learn how to write expressions containing only one term as factors of prime numbers. This helps to simplify the expression by using the rules of indices.

You will also learn how simplify expressions containing a polynomial, by:

- working with each term individually in the polynomial (if there are no variables in the expression); or
- factorising the polynomial, if there are variables in the expression.

Let’s begin by examining the basic principles of indices in more detail.

**Basic principles of indices**

We also refer to an index as an exponent. The plural of index is indices.

If \( n \) is a positive integer, and \( x \) is any real number, then we define \( x^n \) (\( x \) to the power \( n \)) as follows:

\[
x^n = x \times x \times x \cdots \times x \quad (n \text{ factors}).
\]

The following are examples of expressions containing a base and an index:

\[
2^5 = 2 \times 2 \times 2 \times 2 \times 2 = 32  \\
3^4 = 3 \times 3 \times 3 \times 3 = 81  \\
a^7 = a \times a \times a \times a \times a \times a \times a
\]

[base is 2; index is 5]  
[base is 3; index is 4]  
[base is \( a \); index is 7]

Indices have certain laws that can be applied to simplify expressions. Let’s work through these laws of indices to make sure that you are familiar with the laws, and that you know how to apply them.

**The laws of indices**

There are four laws of indices. The four laws relate to the following situations:

- the multiplication of powers that have the same base;
- the division of powers that have the same base;
- indices separated by brackets; and
- the power of a product.

Let’s look at each of these laws in more detail.

**The multiplication of powers that have the same base**

\[
a^m \times a^n = a^{m+n}
\]

As you can see, when multiplying two powers that have the same base, add the indices together.
Let’s look at a few examples of how to apply this law.

**EXAMPLE**

Simplify the following expressions without using a calculator:

1. \( x^4 \times x^5 \)
2. \( 3^4 \times 3^7 \)
3. \( x^{2n} \times x^7 \times x^3 \)

**Answers**

1. \( x^4 \times x^5 = x^{4+5} = x^9 \)
2. \( 3^4 \times 3^7 = 3^{4+7} = 3^{11} \)
3. \( x^{2n} \times x^7 \times x^3 = x^{2n+7+3} = x^{2n+10} \)

The division of powers that have the same base

\[
\frac{a^m}{a^n} = a^{m-n}
\]

When dividing two powers that have the same base, subtract the index of the denominator from the index of the numerator.

Let’s look at a few examples of how to apply this law.

**EXAMPLE**

Simplify the following expressions without using a calculator:

1. \( \frac{x^{12}}{x^3} \)
2. \( \frac{5^7}{5^3} \)
3. \( \frac{x^{3a+2}}{x^{3a-5}} \)
Answers

1. \( \frac{x^{12}}{x^3} = x^{12-3} \)
   
   \[ = x^9 \]

2. \( \frac{5^7}{5^3} = 5^{7-3} \)

   \[ = 5^4 \]

3. \( \frac{x^{3a+2}}{x^{3a-5}} = x^{3a+2-(3a-5)} \)

   \[ = x^{3a+2-3a+5} \]

   \[ = x^7 \]

Indices separated by brackets

\[ \sum \]

\( (a^m)^n = a^{mn} \)

Whenever there are indices separated by brackets, we must multiply these indices together.

Let’s look at a few examples of how to apply this law:

**EXAMPLE**

Simplify the following expressions without using a calculator:

1. \( (x^{15})^3 \)

2. \( (5^3)^{x+1} \)

3. \( (a^{x-4})^{x+4} \)

Answers

1. \( (x^{15})^3 = x^{15 \times 3} \)

   \[ = x^{45} \]

   Alternatively:

   \( (x^{15})^3 = (x^{15}) (x^{15}) (x^{15})^3 \)

   \[ = x^{15+15+15} \]

   \[ = x^{45} \]
2. \((5^3)^{x+1} = (5^3)^{x+1}\)
   \[= 5^{3(x+1)}\]

3. \((a^{x-4})^{x+4} = a^{(x-4)(x+4)}\)
   \[= a^{x^2-16}\]

---

**The power of a product**

\[\sum\]

\((ab)^m = a^m b^m\)

The power \((m)\) of a product \((ab)\) is equal to the product of the powers of the factors. \((a^m b^m)\)

Let’s look at a few examples of how to apply this law:

---

**EXAMPLE**

Simplify the following expressions without using a calculator:

1. \((6)^4\)
2. \((x^4 y^2 z^5)^3\)

**Answers**

1. \((6)^4 = (2 \times 3)^4\)
   \[= (2^4 \times 3^4)\]
   \[= 2^{16} \times 3^{16}\]
   \[= 2^4 \times 3^4\]

2. \((x^4 y^2 z^5)^3 = x^{4 \times 3} y^{2 \times 3} z^{5 \times 3}\)
   \[= x^{12} y^6 z^{15}\]

---

**Additional definitions**

In addition to the basic definition of an index and a base given above, three special circumstances need further explanation and definition. These additional definitions cover situations in which an expression has an index that is:

- zero;
- a negative number; or
- a rational number.
In our earlier definition of an index, we specified that \( n \) must always be a positive integer. If this is the case, then what do the following expressions mean?

- \( a^0 \),
- \( a^{-m} \); and
- \( \frac{m}{a^n} \).

**An expression that has an index of 0**

If \( a \) is any non-zero real number, then \( a^0 \) is always equal to 1.

Examples of expressions that have 0 as an index include:

- \( 100^0 = 1 \)
- \( (a + b)^0 = 1 \)
- \( 7x^0 = 7 \times 1 \)
- \( = 7 \)
- \( (2x)^0 = 1 \)

We can prove that \( a^0 \) is equal to 1, as follows:

\[
\begin{align*}
a^0 &= a^{m-m} \\
&= \frac{a^m}{a^m} \\
&= 1
\end{align*}
\]

The expression \( 0^0 \) is undefined. In other words, \( a \) cannot be equal to 0.

**An expression that has an index that contains a negative number**

If \( n \) is a negative number, then we can rewrite the expression so that \( n \) becomes positive.

If \( a \) is any non-zero real number, then:

\[
a^{-m} = \frac{1}{a^m}
\]

And:

\[
\frac{1}{a^{-m}} = a^m
\]

You can always use this rule to write an expression with a positive index. Let's look at a few examples to help make this clear.
EXAMPLE

Simplify the following expressions without using a calculator:

1. \( 3^{-4} \)
2. \( \frac{2}{a^{-2}} \)
3. \( \frac{m^3}{m^{10}} \)
4. \( -2x^{-3} \)
5. \( \frac{a^{-4}}{5} \)

Answers

1. \( 3^{-4} = \frac{1}{3^4} \)
   = \( \frac{1}{81} \)
2. \( \frac{2}{a^{-2}} = 2a^2 \)
3. \( \frac{m^3}{m^{10}} = m^{3-10} \)
   = \( m^{-7} \)
   = \( \frac{1}{m^7} \)
4. \( -2x^{-3} = \frac{-2}{x^3} \)
5. \( \frac{a^{-4}}{5} = \frac{1}{5a^4} \)

An expression that has an index that is a rational number

The third special case that we indicated earlier is one in which the index of an expression is a rational number, namely \( \frac{m}{n} \). Note that both \( m \) and \( n \) are real numbers, and \( a \) and \( n \) must both be greater than zero (positive values).

\[ \sum a^n = \sqrt[n]{a^m} \quad (m, n \in \mathbb{Z}, \ n > 0, \ a > 0) \]
The \( n \)th root of a number \( b \), written as \( \sqrt[n]{b} \), is a number \( c \) so that \( c^n = b \).

\[
\therefore \sqrt[n]{b} = b^{\frac{1}{n}}
\]

To write \( \sqrt[n]{a^m} \) in exponential form, \( a \) is the base; the index is a fraction where the numerator is the index of \( a \) (which is \( n \)) and the numerator is the root (which is \( m \)). \( \therefore a^{\frac{m}{n}} \).

The square root (\( \sqrt{\,} \)) is the second root of a number. Therefore:

\[
\sqrt{a} = a^{\frac{1}{2}}
\]

The following examples show how to apply this formula.

**EXAMPLE**

Simplify the following expressions without using a calculator:

1. \( \sqrt[3]{125} \)
2. \( \sqrt{12} \)
3. \( \sqrt[2]{2a^2b^3} \)

**Answers**

1. \( \sqrt[3]{125} = \sqrt[3]{5^3} \)
   \[
   = (5)^{\frac{3}{3}}
   = 5^{1}
   = 5
   \]

2. \( \sqrt{12} = \sqrt{4 \times 3} \)
   \[
   = \sqrt{2^2 \times 3^1}
   = (2^{\frac{2}{2}} \times 3^{\frac{1}{2}}
   = 2^{\frac{1}{2}} \times 3^{\frac{1}{2}}
   = 2^1 \times 3^{\frac{1}{2}}
   = 2 \times 3^{\frac{1}{2}}
   \]
3. \( \sqrt[3]{2a^2 b^3} = \frac{1}{2} \times a^{\frac{2}{3}} \times b^{\frac{3}{3}} \)

\( = \frac{1}{2} \times a^{\frac{2}{3}} \times b \), which can also be written as \( = \frac{1}{2} \times a^{\frac{2}{3}} \times b \)

Now, let's do a few examples where we apply all the laws together.

**EXAMPLE**

Simplify the following expressions without using a calculator:

1. \((-2x)^5\)
2. \(- (3k)^4\)
3. \((-1027)^0\)
4. \(5^{-3}\)
5. \((5a)^{-2}\)
6. \(5a^{-2}\)
7. \((8)^{\frac{2}{3}}\)
8. \((4)^{-\frac{3}{2}}\)
9. \((2x^5 y^8)^4\)
10. \(\frac{3^{2n+1}}{3^{2n-1}}\)
11. \((-5a^4 b^6)^2 \times (-2b^3 a^3)^3\)
12. \(\sqrt[4]{81^3}\)

**Answer**

1. \((-2x)^5 = (-2)^5 \times (x)^5\)
   \[= -32x^5\]
2. \(- (3k)^4 = -[(3)^4 \times (k)^4]\)
   \[= -[81 \times k^4]\]
   \[= -81k^4\]
3. \((-1027)^0 = 1\)
4. \[ 5^{-3} = \frac{1}{5^3} = \frac{1}{125} \]

5. \[ (5a)^{-2} = \frac{1}{(5a)^2} \]
   \[ = \frac{1}{(5a) \times (5a)} \quad \text{or} \quad \frac{1}{(5^2) \times (a^2)} \]
   \[ = \frac{1}{25a^2} \]

6. \[ 5a^{-2} = 5 \times \frac{1}{a^2} \]
   \[ = \frac{5}{a^2} \]

7. \[ (8)^{\frac{2}{3}} = \frac{\sqrt[3]{8^2}}{3} \text{ alternatively } \left(2^3\right)^{\frac{2}{3}} \quad \text{(using prime factors)} \]
   \[ = \frac{\sqrt[3]{64}}{2} \quad \text{or} \quad \frac{\sqrt[3]{8^2}}{2} = 2^{3 \times \frac{2}{3}} \]
   \[ = \frac{\sqrt[3]{4^3}}{6} \quad \text{or} \quad \frac{6}{2^3} = 2^2 = 2^2 \]
   \[ = 4 \quad \text{or} \quad = 4 \]

8. \[ (4)^{\frac{3}{2}} = \frac{1}{\left(\frac{3}{2}\right)^2} \]
   \[ = \frac{1}{\sqrt{(4)^3}} \quad \text{or} \quad \frac{1}{\left(2^2\right)^{\frac{3}{2}}} \quad \text{(using prime factors)} \]
   \[ = \frac{1}{\sqrt{64}} = \frac{1}{\left(2^3\right)^{\frac{3}{2}}} \]
   \[ = \frac{1}{8} = \frac{1}{2^3} = \frac{1}{8} \]

9. \[ (2x^5y^4)^4 = 2^4x^{5 \times 4}y^{4 \times 4} \]
   \[ = 16x^{20}y^{32} \]
10. \( \frac{3^{2n+1}}{3^{2n-1}} = 3^{2n+1-(2n-1)} \)
    \[ = 3^{2n+1-2n+1} \]
    \[ = 3^2 \]
    \[ = 9 \]

11. \((-5a^4b^6)^2 \times (-2b^3a^3)^3 = [(-5)^2(a^4)^2(b^6)^2] \times [(-2)^3(a^3)^3(b^3)^3] \)
    \[ = (25a^8b^{12}) \times (-8a^9b^9) \]
    \[ = (25 \times -8) \times (a^{8+9}) \times (b^{12+9}) \]
    \[ = -200a^{17}b^{21} \]

12. \( \sqrt[3]{81^3} = 81^{\frac{3}{4}} \)
    \[ = (3^4)^{\frac{3}{4}} \]
    \[ = 3^3 \]
    \[ = 27 \]

Now work carefully through the following activity to practise what you have learnt about indices.

### Activity 1

Simplify the following expressions (if possible) without using a calculator. Write down all answers with positive indices.

1. \( \frac{5^4}{5^6} \)
2. \( 27^{\frac{2}{3}} \times 4^{\frac{3}{2}} \)
3. \( \left( \frac{625a^4}{256b^8} \right)^{\frac{1}{4}} \)
4. \( \left( x^2 + y^2 \right)^{\frac{1}{2}} \)
Answer

1. \( \frac{5^4}{5^6} = 5^{4-6} \)
   
   \[ = 5^{-2} \]
   
   \[ = \frac{1}{5^2} \]
   
   \[ = \frac{1}{25} \]

2. \( 27 \frac{2}{3} \times 4^2 = \left(3^3\right)^{\frac{2}{3}} \times \left(2^2\right)^{\frac{3}{2}} \) (prime factors)

   \[ = 3^{3 \times \frac{2}{3}} \times 2^{2 \times \frac{3}{2}} \] (apply law 3)

   \[ = 3^2 \times 2^3 \]

   \[ = 9 \times 8 \]

   \[ = 72 \]

3. \( \left( \frac{625a^4}{256b^8} \right)^{\frac{1}{4}} = \left( \frac{5^4a^4}{2^8b^8} \right)^{\frac{1}{4}} \) (Prime factors)

   \[ = \left(5^4\right)^{\frac{1}{4}} \times (a^4)^{\frac{1}{4}} \] (apply law 4)

   \[ = \frac{5^{-1} \times a^{-1}}{2^{-2} \times b^{-2}} \] (apply law 3)

   \[ = \frac{2^2b^2}{5a} \] (write with positive indices)

   \[ = \frac{4b^2}{5a} \]

4. \( (x^2 + y^2)^{\frac{1}{2}} \)

You cannot simplify the expression \( (x^2 + y^2)^{\frac{1}{2}} \). Note that the laws of indices do not make provision for addition and subtraction.
Simplifying expressions consisting of one term

In the first part of this lesson, you learnt about the definitions and laws of indices. In the last two sections of the lesson, we’ll look at using the definitions and laws to simplify expressions that are more complex.

In this section, we focus specifically on expressions consisting of one term only.

Keep the following guidelines in mind when simplifying expressions:

- Convert all fractions or decimals to proper fractions \( \left( \frac{2}{3} \right) \) or improper fractions \( \left( \frac{3}{2} \right) \).
- Always give answers with positive exponents.
- Whenever possible, reduce numbers in the expression to their prime factors, for example, \( 27 = 3^3 \) or \( 4^3 = (2)^3 = 2^6 \).
- Look for like terms when adding or subtracting exponents.
- Look for common factors when adding or subtracting terms.

Let’s work through an example that uses the laws and definitions of indices to simplify expressions containing one term.

**EXAMPLE**

Simplify each of the following expressions:

1. \( \frac{9}{16} \)^{\frac{1}{2}}

2. \( \sqrt{\frac{1}{16^3}} \)

3. \( (5 \frac{1}{16})^{\frac{1}{2}} \)

**Answers**

1. \( \frac{9}{16} \)^{\frac{1}{2}} = \left( \frac{9}{16} \right)^{\frac{3}{2}} \) (convert to improper fractions)

   \[ = \left( \frac{3^2}{2^4} \right)^{\frac{3}{2}} \]

   \[ = \frac{3^3}{2^6} \]

   \[ = \frac{2^6}{3^3} \] (write as positive indices)

   \[ = \frac{64}{27} \]

   \[ = 2 \frac{10}{27} \]
2. \[ \sqrt[4]{\frac{1}{16^3}} = \sqrt[4]{\frac{1}{(2^4)^3}} \]

= \[ \frac{1}{\sqrt[4]{2^{12}}} \]

= \[ \frac{1}{(2^{12})^{\frac{1}{4}}} \]

= \[ \frac{1}{2^3} \]

= \[ \frac{1}{8} \]

3. \[ \left(5 \frac{1}{16}\right)^{-\frac{1}{2}} = \left(\frac{81}{16}\right)^{-\frac{3}{2}} \]

= \[ \left(\left(\frac{3}{2}\right)^4\right)^{-\frac{3}{2}} \]

= \[ \left(\frac{3}{2}\right)^{-6} \]

= \[ \left(\frac{2}{3}\right)^6 \]

= \[ \frac{64}{729} \]

In the final section of this lesson, we'll demonstrate how to simplify exponential expressions.

**Simplification of exponential expressions**

Often, the most important step in the case of exponential expressions is to break up all the bases into powers of prime factors. After that, you can apply the laws of indices by working systematically, one step at a time.

Let’s work through a few examples.
**EXAMPLE**

Simplify the following expressions (without using a calculator) by applying the laws of indices. Write all your answers with positive indices.

1. \[
\frac{9^{x+1} \times 3^{x-4}}{27^{x+1} \times 9^{x-2}}
\]  
   (Hint: The denominator and numerator consist of factors only)

2. \[
\frac{5 \times 2^x - 4 \times 2^{x+2}}{2^{x+2} - 2^{x+1}}
\]  
   (Hint: The denominator and numerator consist of terms)

3. \[
\frac{2^{x+1} \times 3^{2x-3}}{18^x}
\]  
   (Hint: The denominator and numerator consist of factors only)

**Answers**

1. \[
\frac{9^{x+1} \times 3^{x-4}}{27^{x+1} \times 9^{x-2}} = \frac{(3^2)^{x+1} \times 3^{x-4}}{(3^3)^{x+1} \times (3^2)^{x-2}}
\]
   \[
   = \frac{3^{2x+2} \times 3^{x-4}}{3^{3x+3} \times 3^{2x-4}}
\]
   (apply law 3)
   \[
   = 3^{(2x+2)+(x-4)-(3x+3)-(2x-4)}
\]
   (apply laws 1 and 2)
   \[
   = 3^{-1}
\]
   \[
   = \frac{1}{3}
\]
   (write as a positive index)

2. \[
\frac{5 \times 2^x - 4 \times 2^{x+2}}{2^{x+2} - 2^{x+1}} = \frac{5 \times 2^x - 4 \times 2^x \times 2^2}{2^x \times 2^2 - 2^x \times 2^1}
\]
   (apply law 1 in reverse order)
   \[
   = \frac{2^x(5 - 4 \times 2^2)}{2^x(2^2 - 2)}
\]
   (factorise)
   \[
   = \frac{5 - 16}{4 - 2}
\]
   (cancel equal factors)
   \[
   = \frac{-11}{-2}
\]
   \[
   = \frac{11}{2} \text{ or } 5 \frac{1}{2}
\]
3. \[ \frac{2^{x+1} \cdot 3^{2x-3}}{18^x} = \frac{2^x \cdot 2^1 \cdot 3^{2x} \cdot 2^{-3}}{(2 \cdot 3^2)^x} \]

\[ = \frac{2^x \cdot 2^1 \cdot 3^{2x} \cdot 2^{-3}}{2^x \cdot 3^{2x}} \]

\[ = 2^{x-1} \cdot 3^{2x-2x} \]

\[ = 2^{-2}. \]

\[ = \frac{1}{4} \]

Now work carefully through the following activity to practise what you have learnt about simplifying expressions without using a calculator.

---

**Activity 2**

Simplify the following expressions without using a calculator. Give all answers with positive indices.

1. \( \left(27x^{-6}y^9\right)^{\frac{2}{3}} \)
2. \( \sqrt[3]{\frac{3^a \cdot 9^{a+1}}{27^{a+2}}} \)
3. \( \frac{3 \cdot 2^{a+1} - 2^{a-2} + 2^a}{4 \cdot 2^{a-3}} \)
4. \( \frac{x + y}{x^{-1} + y^{-1}} \)

**Answers**

1. \( \left(27x^{-6}y^9\right)^{\frac{2}{3}} = \left(3^3x^{-6}y^9\right)^{\frac{2}{3}} \)

\[ = \left(3^3\right)^{\frac{2}{3}} \times \left(x^{-6}\right)^{\frac{2}{3}} \times \left(y^9\right)^{\frac{2}{3}} \]

\[ = 3^{-2} \times x^4 \times y^6 \]

\[ = \frac{x^4}{9y^6} \]
2. \[ \sqrt[4]{\frac{3^a \times 9^{a+1}}{27^{a+2}}} = \left( \frac{3^a \times 9^{a+1}}{27^{a+2}} \right)^{\frac{1}{4}} \]

\[ = \left( \frac{3^a \times 3^{2a+2}}{3^{3a+6}} \right)^{\frac{1}{4}} \] (Write in exponential form)

\[ = \left( 3^{a+2a+2-(3a+6)} \right)^{\frac{1}{4}} \] (Prime factors; apply law 3)

\[ = \left( 3^{-1} \right)^{\frac{1}{4}} \] (Apply laws 1 and 2)

\[ = 3^{-1} \] (Apply law 3)

\[ = \frac{1}{3} \] (Write with a positive index)

3. \[ \frac{3 \times 2^{a+1} - 2^{a-2} + 2^a}{4 \times 2^{a-3}} = \frac{3 \times 2^a \times 2 - 2^a \times 2^{-2} + 2^a}{4 \times 2^a \times 2^{-3}} \] (law 1 in reverse order)

\[ = \frac{2^a (3 \times 2 - 2^{-2} + 1)}{2^a \times 2^2 \times 2^{-3}} \] (take out common factor of \( 2^a \) in numerator)

\[ = \frac{(3 \times 2 - 2^{-2} + 1)}{2^2 \times 2^{-3}} \]

\[ = \frac{(6 - \frac{1}{2} + 1)}{2 \times \frac{1}{8}} \]

\[ = \frac{(6 - \frac{1}{4} + 1)}{4 \times \frac{1}{8}} \]

\[ = \frac{6 - \frac{3}{4}}{\frac{1}{2}} \]

\[ = \frac{6}{1} \times \frac{1}{2} \]

\[ = \frac{6}{2} \]

\[ = \frac{3}{2} \]

\[ = \frac{27}{2} \times 2 \]

\[ = \frac{27}{2} \]

\[ = 13 \frac{1}{2} \]
Summary

In Lesson 1, you learnt about indices, which was revision of Grade 11 topics. The first part of the lesson introduced you to the basic concepts of indices, including:

- the four definitions of indices; and
- the four laws of indices.

The second part of the lesson focused on applying the laws and definitions to simplifying mathematical expressions containing indices. Here, you learnt how to simplify expressions containing:

- only factors;
- only one term; and
- expressions containing polynomials (more than one term).

Remember to use the following guidelines when deciding on how to simplify an expression:

- Only work with proper fractions $\frac{2}{3}$ or improper fractions $\frac{3}{2}$. In other words, always convert decimal values or mixed numbers to fractions before trying to simplify the expression.
- Always give answers with positive exponents.
- If the expression contains one or more terms, then you may first need to factorise the expression.
- Whenever possible, reduce numbers in the expression to their prime factors. This may often help you to simplify an expression by adding or subtracting exponents for values that have the same base. You may also find that both the numerator and denominator have factors in common, which you can then cancel out.
- Look for like terms when adding or subtracting exponents.
- Look for common factors when adding and subtracting terms.
Self-assessment Questions 1

Test your knowledge of this lesson by completing the self-assessment questions below. When you answer the questions, don’t look at the suggested answers that we give straight away. Look at them only after you’ve written your answers down, and then compare your answers with our answers. (Complete the assessment first. If you are not satisfied, redo the assessment.)

Simplify the following expressions without using a calculator.

1. \( \left( \frac{8}{125} \right)^{\frac{2}{3}} \)
2. \( (0.0016)^{\frac{1}{2}} \)
3. \( (32x^{-5}y^{25})^{\frac{2}{5}} \)
4. \( (2\frac{1}{4})^{\frac{1}{4}} \times \sqrt[3]{\frac{3}{8}} \)
5. \( \left( \frac{16a^{3}b^{-4}}{9ab^{2}} \right)^{\frac{1}{2}} \)
6. \( \frac{15^{8} \times 4^{7}}{6^{7} \times 10^{6}} \)
7. \( \frac{5^{a+1} \times (4^{a})^{3}}{2^{5a+2} \times 10^{a-1}} \)
8. \( \frac{3^{2x+3} - 3^{2x-1}}{3^{2x-2}} \)
9. \( \frac{5^{p+1} - 5^{p}}{5^{p} + 5^{p-1}} \)

Suggested answers to Self-assessment Questions 1

1. \( \left( \frac{8}{125} \right)^{\frac{2}{3}} = \left( \frac{2^{3}}{5^{3}} \right)^{\frac{2}{3}} \)
   \[ = \left( \frac{2^{3}}{5^{3}} \right)^{\frac{2}{3}} \]
   \[ = \frac{2^{2}}{5^{2}} \]
   \[ = 2^{-2} \]
   \[ = \frac{25}{4} \]
2. \( (0.0016)^{\frac{1}{4}} = \left( \frac{16}{10000} \right)^{\frac{1}{4}} \)
\[ = \left( \frac{2}{10} \right)^{\frac{1}{4}} \]
\[ = \left( \frac{2}{10} \right)^{4 \cdot \frac{1}{4}} \]
\[ = \frac{2}{10} \]
\[ = \frac{1}{5} \]

3. \( (32x^{-5}y^{25})^{\frac{2}{5}} = (2^5x^{-5}y^{25})^{\frac{2}{5}} \)
\[ = (2^5)^{\frac{2}{5}} \cdot (x^{-5})^{\frac{2}{5}} \cdot (y^{25})^{\frac{2}{5}} \]
\[ = 2^{-2} \cdot x^2 \cdot y^{10} \]
\[ = \frac{x^2}{2^2 \cdot y^{10}} \]
\[ = \frac{x^2}{4y^{10}} \]

4. \( (2 \frac{1}{2})^{\frac{1}{4}} \times \sqrt[3]{\frac{3}{8}} = \left( \frac{9}{4} \right)^{\frac{1}{4}} \times \left( \frac{3}{8} \right)^{\frac{1}{3}} \)
\[ = \left( \frac{3^2}{2^2} \right)^{\frac{1}{4}} \times \left( \frac{3}{2^3} \right)^{\frac{1}{3}} \]
\[ = \frac{3^{\frac{1}{2}}}{2^{-\frac{1}{2}}} \times \frac{3^{\frac{1}{3}}}{2^{\frac{3}{3}}} \]
If we look at the top line: \( 3^{\frac{1}{2}} \times 3^{\frac{1}{3}} = 3^{\frac{1}{2} + \frac{1}{3}} = 3^{\frac{1}{2} \cdot \frac{3}{2}} \)
\[ = 3^0 \]
The denominators simplify: \( 2^{-\frac{1}{2}} \times 2^{\frac{3}{2}} = 2^{1} \)
\[ = \frac{1}{2} \]
5. \[ \left( \frac{16a^3b^{-4}}{9ab^2} \right)^{\frac{1}{2}} = \left( \frac{2^4a^3b^{-4}}{3^2ab^2} \right)^{\frac{1}{2}} \]

\[ = 2^2 \frac{a^3b^{-2}}{3^1a^3b} \]

Simplify the factors in \( a \): \[ \frac{a^3}{a^2} = a^\frac{3}{2} = a\frac{1}{2} = a \]

\[ = \frac{4}{3} \times ab^{-3} \]

\[ = \frac{4a}{3b^3} \]

6. \[ \frac{15^8 \times 4^7}{6^7 \times 10^6} = \frac{(3 \times 5)^8 \times (2^2)^7}{(2 \times 3)^7 \times (2 \times 5)^6} \]

\[ = \frac{3^8 \times 5^8 \times 2^{14}}{2^7 \times 3^7 \times 2^6 \times 5^6} \]

\[ = 2^{14-7-6} \times 3^{8-7} \times 5^{8-6} \]

\[ = 2^1 \times 3^1 \times 5^2 \]

\[ = 2 \times 3 \times 25 \]

\[ = 150 \]

7. \[ \frac{5^{a+1} \times (4^a)^3}{2^{5a+2} \times 10^{a-1}} = \frac{5^{a+1} \times 2^{6a}}{2^{5a+2} \times 2^{a-1} \times 5^{a-1}} \]

Simplify: \( (4^a)^3 = (2^{2a})^3 = 2^{6a} \)

\[ = 5^{a+1-a+1} \times 2^{6a-5a-2-a+1} \]

\[ = 2^{-1} \times 5^2 \]

\[ = \frac{25}{2} \]

\[ = 12 \frac{1}{2} \]

(Note that terms are separated by a negative sign, so we cannot add the index. We take out a common factor.)
8. \[
\frac{3^{2x+3} - 3^{2x-1}}{3^{2x-2}} = \frac{3^{2x} \times 3^3 - 3^{2x} \times 3^{-1}}{3^{2x} \times 3^{-2}} \\
= \frac{3^{2x}(3^3 - 3^{-1})}{3^{2x} \times 3^{-2}} \\
= \frac{27 - \frac{1}{3}}{\frac{1}{9}} \\
= \frac{81 - 1}{3} \\
= \frac{80}{3} \times \frac{9}{1} \\
= 80 \times 3 \\
= 240
\]

9. \[
\frac{5^{p+1} - 5^p}{5^p + 5^{p-1}} = \frac{5^p(5 - 1)}{5^p(1 + \frac{1}{5})} \\
= \frac{4}{\frac{6}{5}} \\
= 4 \times \frac{5}{6} \\
= \frac{10}{3}
\]

**Check your competence**

Now that you have worked through this lesson, please check that you can perform the tasks below:

- I understand that not all numbers are real.
- I can simplify expressions using the laws of exponents for rational exponents.
- I can solve non-routine, unseen problems.

**The next lesson**

If you are sure that you understand the contents covered in this lesson, and have achieved your Assessment Standards, start Lesson 2.
Learning Outcomes for Lesson 2

After you have worked through Lesson 2, you should be able to do the following:

<table>
<thead>
<tr>
<th>LO</th>
<th>Assessment Standard</th>
</tr>
</thead>
<tbody>
<tr>
<td>LO 1</td>
<td>AS 11.1.1 Understand that not all numbers are real. (This requires the recognition, but not the study of non-real numbers)</td>
</tr>
<tr>
<td></td>
<td>AS 11.1.2 (a) Simplify expressions using the laws of exponents for rational exponents.</td>
</tr>
<tr>
<td></td>
<td>AS 11.1.6 Solve non-routine, unseen problems.</td>
</tr>
<tr>
<td></td>
<td>AS 12.1.6 Solve non-routine, unseen problems.</td>
</tr>
</tbody>
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We would like to help you to remain in complete control of your studies. So we give you an opportunity to check your competence at the end of this lesson.

Introduction

In Lesson 2, we'll discuss how to solve exponential equations. In particular, we'll look at two types of the following form:

- equations with the unknown in the base, for example \(4^{\frac{3}{x}} = 16\); and
- equations with the unknown in the index, for example \(3^x = 81\).

Equations with the unknown in the base

In this section, we'll deal with the first type of exponential equations, namely equations with the unknown in the base, for example \(4^{\frac{3}{x}} = 16\).

We will then work through some examples. Finally, we will end the lesson with an activity.

Let’s begin!

To solve an equation, we say, for example:

\(x = 2\)

or:

\(a = -5\)

We note that both indices of the unknown is 1. Therefore, the aim is to get the index of the unknown equal 1.
To achieve this, we multiply the index by the opposite of the index of the unknown. The opposite of the index is also known as the reciprocal of the index of the unknown.

So, if we multiply the index with its opposite, we will get 1.

According to the rules of mathematics, if we multiply the left-hand side (LHS) of the equation with the opposite of its index, then we also need to multiply the right-hand side (RHS) of the equation with that same index.

For example, the opposite of the value 2 is $\frac{1}{2}$.

Similarly, the opposite of $\frac{1}{2}$ is 2 (also called the reciprocal or inverse.)

The opposite of $\frac{3}{5}$ is $\frac{5}{3}$.

To check if you determined the correct opposite of an index, the product of the index and its opposite are always equal to 1. For example:

$$\frac{3}{5} \times \frac{5}{3} = 1$$

Remember, all indices that are written as improper fractions must be converted to proper fractions.

Now let's work through a few examples to see how this helps us to solve exponential equations that have an unknown in the base.

**EXAMPLE**

Solve for $x$ without using a calculator:

1. $3 \times \frac{3}{4} = 64$

2. $27 \times \frac{3}{2} = 8$

**Answers**

1. Here, the unknown is $x$, and the index of the unknown is $\frac{3}{4}$.

Therefore, the opposite of the unknown is $\frac{4}{3}$. 
Therefore, we have:

\[ x^{\frac{3}{3}} = 64 \]

\[ \left( x^{\frac{3}{3}} \right)^{\frac{4}{3}} = (64)^{\frac{4}{3}} \quad \text{(raise both sides to the power of } \frac{4}{3} \text{)} \]

\[ x = (64)^{\frac{4}{3}} \quad \text{(apply law 3 on the LHS)} \]

\[ = \left( 2^6 \right)^{\frac{4}{3}} \quad \text{(prime factor)} \]

\[ = 2^8 \quad \text{(apply law 3 on the RHS)} \]

\[ = 256 \]

2. Here, the unknown is \( x \), and the index of the unknown is \( -\frac{3}{2} \).

Therefore, the reciprocal of the unknown is \( -\frac{2}{3} \).

Therefore, we have:

\[ 27x^{-\frac{3}{2}} = 8 \]

\[ x^{-\frac{3}{2}} = \frac{8}{27} \quad \text{(isolate the unknown)} \]

\[ \left( x^{-\frac{3}{2}} \right)^{-\frac{2}{3}} = \left( \frac{8}{27} \right)^{\frac{2}{3}} \quad \text{(use the reciprocal of the index)} \]

\[ x = \left( \frac{27}{8} \right)^{\frac{2}{3}} \quad \text{(write as a positive index)} \]

\[ = \left( \frac{3^3}{2^3} \right)^{\frac{2}{3}} \quad \text{(use prime factors)} \]

\[ = \frac{3^2}{2^2} \]

\[ = \frac{9}{4} \]

Now work carefully through the following activity to practise what you have learnt about solving equations with an unknown in the base.
Activity 3

Solve for $x$ without using a calculator:

1. $8x^{\frac{1}{3}} - 32 = 0$
2. $2 - 16x^{-\frac{2}{3}} = 0$
3. $7\left(\frac{1}{x^3} + 1\right)^{\frac{1}{2}} = 21$

Answer

1. Here, the unknown is $x$, and the index of the unknown is $\frac{1}{3}$.

Therefore, the reciprocal of the unknown is $\frac{3}{1}$, or 3.

Therefore, we have:

$$8x^{\frac{1}{3}} - 32 = 0$$

$$8x^{\frac{1}{3}} = 32 \left(\frac{32}{8} = 4\right)$$

$$x^{\frac{1}{3}} = 4 \quad \text{(isolate the unknown)}$$

$$\left(x^{\frac{1}{3}}\right)^3 = (4)^3 \quad \text{or} \quad \left(x^{\frac{1}{3}}\right)^3 = (2^2)^3$$

$$x = 2^6$$

$$x = 64$$

2. Here, the unknown is $x$, and the index of the unknown is $-\frac{3}{2}$.

Therefore, the reciprocal of the unknown is $-\frac{2}{3}$.

Therefore, we have:

$$2 - 16x^{-\frac{3}{2}} = 0$$

$$2 = 16x^{-\frac{3}{2}} \left(\frac{2}{16} = \frac{1}{8}\right)$$

$$x^{-\frac{3}{2}} = \frac{1}{8} \quad \text{(isolate the unknown)}$$

$$\left(x^{-\frac{3}{2}}\right)^{-\frac{2}{3}} = \left(\frac{1}{8}\right)^{\frac{2}{3}}$$
Equations with the unknown in the exponent

In the previous section, you learnt how to solve equations that had the unknown in the base. In this section, we’ll solve what is known as exponential equations. The key to solving exponential equations lies in the following principle:

If \( a^m = a^n \), then \( m = n \), provided that \( a \neq 0 \) and \( a \neq \pm 1 \).

In other words, if the bases are equal, then the exponents of the base should also be equal.

Therefore, to solve exponential equations, you’ll need to follow these steps:

- **Step 1:** Rewrite the bases on the LHS and the RHS of the equation so that they are the same.
- **Step 2:** If the base on LHS of the equation is equal to the base on the RHS of the equation, the bases can be ignored such that the indices are left over. Then we solve the indices using methods of equations.
- **Step 3:** Always check the calculated solution in the original equation to ensure the solution is valid.
The most effective way to make the bases on both sides of the equation equal is to express the values using prime factors.

Let’s work through a few examples to see how to solve exponential equations with the unknown in the exponent.

**EXAMPLE**

Solve for $x$ without using a calculator:

1. $2^x = 16$
2. $2^{x(x-2)} = 8$
3. $10^{2x - 5} = 0.1$
4. $5(2^x) + 5^{-1}(2^x) = \frac{13}{5}$
5. $4^x - 5(2^x) + 4 = 0$
6. $7^x - 8 + 7^{1-x} = 0$

**Answers**

1. $2^x = 16$
   
   $2^x = 2^4$

   $\therefore x = 4$

2. $2^{x(x-2)} = 8$

   $2^{x(x-2)} = 2^3$

   Therefore:

   $x(x-2) = 3$

   $x^2 - 2x - 3 = 0$

   $(x + 1)(x - 3) = 0$

   Therefore:

   $x = -1$ or $x = 3$

   To check if the solutions are valid, we substitute the 2 values of $x$ in the original equation to check if the LHS = RHS. If the LHS = RHS, then the solutions are valid.
Therefore, we have:

For $x = -1$:

\[
\text{LHS: } 2^{-1(-3)} = 2^3 = 8 = \text{RHS}
\]

And for $x = 3$:

\[
\text{LHS: } 2^{3(1)} = 2^3 = 8 = \text{RHS}
\]

Therefore, both solutions are valid.

3. \(10^{2x - 5} = 0,1\)

The first step is to express 0,1 as a fraction and then to try to write the fraction as a base of a whole number.

Now \(0,1 = \frac{1}{10}\). As a base of a whole number \(\frac{1}{10}\) becomes \(10^{-1}\).

Therefore, the original equation can be rewritten as:

\(10^{2x - 5} = 10^{-1}\)

Therefore:

\[2x - 5 = -1\]

\[2x = 4\]

\[x = 2\]

Again, let’s check the validity of the solution:

For $x = 2$:

\[
\text{LHS: } 10^{2(2) - 5} = 10^{4-5} = 10^{-1} = \frac{1}{10} = \text{RHS}
\]

Therefore, the solution $x = 2$ is valid.
4. \[5(2^1) + 5^{-1}(2^1) = \frac{13}{5}\]

\[\therefore 2^x(5 + 5^{-1}) = \frac{13}{5}\]  
(take out a common factor of \(2^x\))

\[\therefore 2^x \left(\frac{5 + 1}{5}\right) = \frac{13}{5}\]  
(simplify the brackets)

\[\therefore 2^x \left(\frac{26}{5}\right) = \frac{13}{5}\]  
(simplify the brackets)

\[\therefore 2^x \left(\frac{26}{5}\right) \times \left(\frac{5}{26}\right) = \frac{13}{5} \times \left(\frac{5}{26}\right)\]  
(multiply both sides by \(\frac{5}{26}\))

\[\therefore 2^x = \frac{1}{2}\]

\[\therefore 2^x = 2^{-1}\]

\[\therefore x = -1\]

Let’s check the validity of the solution in the original equation:

\[
\text{LHS: } 5(2^1) + 5^{-1}(2^1) = \frac{5}{2} + \frac{1}{5 \times 2}
\]

\[
= \frac{5}{2} + \frac{1}{10}
\]

\[
= \frac{25 + 1}{10}
\]

\[
= \frac{26}{10}
\]

\[
= \frac{13}{5}
\]

\[= \text{RHS}\]

Therefore, the solution \(x = -1\) is valid.

5. \[4^x - 5(2^x) + 4 = 0\]

\[(4^x) - 5(2^2) + 4 = 0\]

\[(2^{2x}) - 5(2^x) + 4 = 0\]

\[(2^x)^2 - 5(2^x) + 4 = 0\]

If we substitute \(2^x = a\), then the equation becomes:

\[a^2 - 5a + 4 = 0\]

\[(a - 4)(a - 1) = 0\]

\[\therefore a = 4\]

or

\[a = 1\]
Remember, we substituted $2^x = a$. So now we substitute $2^x = a$ back in the solutions $a = 4$ or $a = 1$.

Also note that we need to solve for $x$ and not $a$. So, if $a = 4$, our first solution becomes:

$$2^x = 4$$

$$\therefore 2^x = 2^2$$

$$\therefore x = 2$$

And for $a = 1$, we have:

$$2^x = 1$$

$$\therefore 2^x = 2^0$$

$$\therefore x = 0$$

Finally, we only need to check the validity of the solutions by substituting our solution for $x$ in the original equation and check if the LHS = RHS.

So, for $x = 2$, we have:

LHS: $4^x - 5(2^x) + 4 = 4^2 - 5(2) + 4$

$$= 16 - 20 + 4$$

$$= 0$$

$$= \text{RHS}$$

And, for $x = 0$, we have:

LHS: $4^x - 5(2^x) + 4 = 4^0 - 5(2^0) + 4$

$$= 1 - 5(1) + 4$$

$$= 1 - 5 + 4$$

$$= 0$$

$$= \text{RHS}$$

Therefore, both solutions for $x$ are valid.

6. $7^x - 8 + 7^{1-x} = 0$

$$7^x - 8 + \frac{7}{7^x} = 0$$

Let $7^x = a$, then our equation becomes:

$$a - 8 + \frac{7}{a} = 0$$
If we multiply the equation by \( a \), we get
\[
a^2 - 8a + 7 = 0
\]
\[
\therefore (a - 7)(a - 1) = 0
\]
\[
\therefore a = 7 \text{ or } a = 1
\]
So now we substitute \( 7^x = a \) back in the solutions \( a = 7 \) or \( a = 1 \).

So if \( a = 7 \), then our first solution becomes:
\[
7^x = 7
\]
\[
\therefore 7^x = 7^1
\]
\[
\therefore x = 1
\]
And for \( a = 1 \), we have:
\[
7^x = 1
\]
\[
\therefore 7^x = 7^0
\]
\[
\therefore x = 0
\]
We only need to check the validity of the solutions by substituting our solution for \( x \) in the original equation and check if the LHS = RHS.

So, for \( x = 1 \), we have:
\[
\text{LHS: } 7^x - 8 + 7^{1-x} = 7^1 - 8 + 7^{1-1}
\]
\[
= 7^1 - 8 + 7^0
\]
\[
= 7 - 8 + 1
\]
\[
= 0
\]
\[
= \text{RHS}
\]
And for \( x = 0 \), we have:
\[
7^x - 8 + 7^{1-x} = 7^0 - 8 + 7^{1-0}
\]
\[
= 1 - 8 + 7
\]
\[
= 0
\]
Therefore, both solutions for \( x \) are valid.

Now work carefully through the following activity to practise what you have learnt about solving exponential equations.
Activity 4

Solve for \( x \) without using a calculator:

1. \( 8^{2x-3} = 128 \)
2. \( (5^x)^{-1} = 25 \)
3. \( 3 \times 2^x = 0,375 \)
4. \( 2 \times 2^x - 8 \times 2^{-x} = 12 \)
5. \( 2 \times 2^x - 15 = 8 \times 2^{-x} \)

Answers

1. \( 8^{2x-3} = 128 \)

\[
(2^3)^{2x-3} = 2^7 \\
2^{6x-9} = 2^7 \\
\therefore 6x - 9 = 7 \\
\therefore 6x = 16 \\
\therefore x = \frac{16}{6} \text{ (or } \frac{8}{3} \text{)}
\]

(if the bases are the same, the indices are equal)

2. \( (5^x)^{-1} = 25 \)

\( 5^{x^2 - x} = 5^2 \)

Therefore:

\[
x^2 - x = 2 \\
x^2 - x - 2 = 0 \\
(x-2)(x+1) = 0 \\
x = 2 \text{ or } x = -1
\]

You need to check these possible solutions in the original equation.

Therefore, for \( x = 2 \), you have:

LHS: \( (5^x)^{-1} = 25 \)

\[
\left(5^2\right)^{2-1} \\
= (5^2)^1 \\
= 25^1 \\
= 27 \\
= \text{RHS}
\]

Therefore, \( x = 2 \) is valid.
And, for \( x = -1 \), you have:

LHS: \((5^x)^{-1} = \left(5^{-1}\right)^{-1} = 5^2 = 25\)

Therefore, \( x = -1 \) is valid, and both solutions for \( x \) are valid.

3. \(3 \times 2^x = 0.375\)
   \(2^x = 0.125\) (divide both sides by 3)
   \(2^x = \frac{1}{8}\)
   \(0.125 = \frac{125}{1000} = \frac{1}{8}\)
   \(2^x = 2^{-3}\)

Therefore, \( x = -3 \).

4. \(2 \times 2^x - 8 \times 2^{-4} = 12\)
   \(2^x(2 - 8 \times 2^{-4}) = 12\) (remove common factor)
   \(2 - 8 \times 2^{-4} = 2 - 8 \times \frac{1}{2^4} = 2 - 8 \times \frac{1}{16} = 2 - \frac{1}{2} = \frac{3}{2}\)
   \(\therefore 2^x \left(\frac{3}{2}\right) = 12\)
   \(\therefore 2^x \times \frac{3}{2} \times \frac{2}{3} = 12 \times \frac{2}{3}\) (multiply both sides by \(\frac{2}{3}\))
   \(\therefore 2^x = 8\)
   \(2^x = 2^3\)

Therefore, \( x = 3 \).

5. \(2 \times 2^x - 15 = 8 \times 2^{-x}\)
   \(2 \times 2^x - 8 \times 2^{-x} = 15\)
   \(2 \times 2^x - \frac{8}{2^x} = 15\)
   \(2 \times (2^x)^2 - 8 = 15 \times (2^x)\) (multiply both sides by \(2^x\))

\[2 \times (2^x)^2 - 15 \times (2^x) - 8 = 0\]

Put \(2^x = k\)
\[2^{2x} = k^2\]
\[2k^2 - 15k - 8 = 0\]
\[(2k + 1)(k - 8) = 0\]
\[k = -\frac{1}{2} \quad \text{or} \quad k = 8\]
Therefore, there are two possible solutions to the equation, both of which are exponential equations. The first solution is:

\[2^x = 8\]
\[2^x = 2^3\]

Therefore:
\[x = 3\]

The second solution is:
\[2^x = -2\]

The second solution is invalid, because \(2^x > 0\) for all values of \(x\). Remember that the index must always be a positive number.

Therefore, the final solution to the original equation is \(x = 3\) only.

---

**Summary**

In Lesson 2, we examined how to solve equations that contain indices. You learnt how to solve two types of equations, namely:

- equations that contain an unknown in the base; and
- equations that contain an unknown in the index.

You also learnt that equations that have an unknown in the index (or exponent) are also known as *exponential equations*.

Sometime more than one law can get you to your answer. Do the one that is easier for you.

We solve equations that contain an unknown in the *base* by raising both sides of the equation to the reciprocal of the index of the unknown.

We solve equations that contain the unknown in the *index* by applying the principle that if \(a^m = a^n\), then \(m = n\), provided that \(a \neq 0\) and \(a \neq \pm 1\).

---

**Self-assessment Questions 2**

Test your knowledge of this lesson by completing the self-assessment questions below. When you answer the questions, don’t look at the suggested answers that we give straight away. Look at them only after you’ve written your answers down, and then compare your answers with our answers.

Solve for \(x\) without using a calculator:

1. \(3x^{-\frac{1}{2}} = \sqrt{3}\)
2. \(2x^{-\frac{3}{2}} - 54 = 0\)
3. \[5x^\frac{2}{3} - 6x^\frac{1}{3} + 1 = 0\]

4. \[8\left(\frac{1}{2}\right)^{x-2} = 1\]

5. \[5^{2-x} = \frac{1}{2}(5^x + 5)\]

**Suggested answers to Self-assessment Questions 2**

1. \[3x^{\frac{1}{2}} = \sqrt{3}\]
   \[\left(x^{\frac{1}{2}}\right)^2 = \left(\frac{\sqrt{3}}{3}\right)^2\]
   \[x = \left(\frac{3}{\sqrt{3}}\right)^2 = \frac{9}{3} = 3\]

2. \[2x^{-\frac{3}{2}} - 54 = 0\]
   \[2x^{-\frac{3}{2}} = 54\]
   \[x^{-\frac{3}{2}} = 27\]
   \[\left(x^{-\frac{3}{2}}\right)^\frac{2}{3} = (27)^{-\frac{2}{3}}\]
   \[x = (27)^{-\frac{2}{3}} = \left(3^3\right)^{-\frac{2}{3}} = 3^{-2} = \frac{1}{9}\]
3. \[ 5x^{\frac{2}{3}} - 6x^{\frac{1}{3}} + 1 = 0 \]

\[
\left( \frac{1}{5x^{\frac{1}{3}}} - 1 \right) \left( \frac{1}{x^{\frac{1}{3}}} - 1 \right) = 0
\]

Put \( k = x^{\frac{1}{3}} \)

\[ k^2 = x^{\frac{2}{3}} \]

\[ 5k^2 - 6k + 1 = 0 \]

\[ (5k - 1)(k - 1) = 0 \]

\[ 5k - 1 = 0 \quad \text{or} \quad k - 1 = 0 \]

\[ 5k = 1 \quad \text{or} \quad k = 1 \]

Therefore, there are two possible solutions to the equation. The first possible solution is:

\[ x^{\frac{1}{3}} - \frac{1}{5} = 0 \]

\[ \left( \frac{1}{x^{\frac{1}{3}}} \right)^3 = \left( \frac{1}{5} \right)^3 \]

\[ x = \frac{1}{125} \]

The second possible solution is:

\[ x^{\frac{1}{3}} - 1 = 0 \]

\[ \left( \frac{1}{x^{\frac{1}{3}}} \right)^3 = (1)^3 \]

\[ x = 1 \]

4. \[ 8\left( \frac{1}{2} \right)^{x-2} = 1 \]

\[ \left( \frac{1}{2} \right)^{x-2} = \frac{1}{8} \]

\[ = \left( \frac{1}{2} \right)^3 \]

Therefore:

\[ x - 2 = 3 \]

\[ (\text{if the bases are the same, the indices are equal}) \]

\[ x = 5 \]
5. \[5^{2-x} = \frac{1}{2}(5^x + 5)\]

\[5^2 \times 5^{-x} = \frac{1}{2} \times 5^x + \frac{1}{2} \times 5\]

\[
\frac{5^2}{5^x} = \frac{5^x \times 5}{2} + \frac{5}{2}
\]

\[5^2 = \frac{(5^x)^2}{2} + \frac{5}{2}(5^x) \quad \text{(multiply each term by } 5^x)\]

\[25 = \frac{(5^x)^2}{2} + \frac{5}{2}(5^x)\]

\[50 = (5^x)^2 + 5(5^x) \quad \text{(multiply each term by } 2)\]

If you let \(a = 5^x\)

\[50 = a^2 + 5a\]

\[a^2 + 5a - 50 = 0\]

\[(a - 5)(a + 10)\]

\[a = 5 \text{ or } a = -10\]

Therefore, there are two possible solutions.

The first possible solution is:

\[5^x = 5\]

\[5^x = 5^1\]

Therefore, \(x = 1\).

The second possible solution is:

\[5^x = -10\]

Since \(x\) must be positive, there is no value of \(x\) that will satisfy the second solution.

Therefore, the only valid solution to the equation is \(x = 1\).
Check your competence

Now that you have worked through this lesson, please check that you can perform the tasks below:

- I understand that not all numbers are real.
- I can simplify expressions using the laws of exponents for rational exponents.
- I can solve non-routine, unseen problems.

The next lesson

If you are sure that you understand the contents covered in this lesson, and have achieved your Assessment Standards, start Lesson 3.
LO 1

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<td>Simplify expressions using the laws of exponents for rational exponents.</td>
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As before, we would like to help you to remain in complete control of your studies. So, once again, we provide a checkbox at the end of this lesson.

Introduction

Lesson 3 is all about surds.

By the end of this lesson, you’ll know exactly what a surd is, and how to work with surds in mathematical calculations.

There are two main sections to the lesson, namely:

- the basic principles of surds; and
- calculations with surds.

Let’s begin!

The basic principles of surds

Before we can define a surd, we need to know the difference between a rational number and an irrational number.

A rational number is any real number that we can express as a ratio of two integers. Examples of rational numbers include \( \frac{1}{2} \) and \( \frac{5}{1} \).
An **irrational number** is any real number that is not a rational number. An example of an irrational number is $\sqrt{2}$. Most real numbers are irrational numbers.

Now, let’s define a **surd**.

A **surd** is an irrational number in the form $\sqrt[n]{m}$. We sometimes also refer to surds as **radicals**.

Therefore, $\frac{3}{\sqrt{8}}$ is not a surd, because $\frac{3}{\sqrt{8}} = 2$, which is a rational number. However, $\sqrt{8}$ is a surd.

Finally, recall from Lesson 1 that $\sqrt[n]{m} = \frac{m}{n}$. Now let’s have a look at the four surd laws.

**Surd law 1**

$$\sqrt[n]{a} \times \sqrt[n]{b} = \sqrt[n]{ab}, \ (a, b \in \mathbb{Z}^+, \ n \in \mathbb{N})$$

For example:

$$\sqrt{2} \times \sqrt{2} = \sqrt{4}$$

$$= 2$$

And:

$$\sqrt[3]{27} \times \sqrt[3]{9} = \sqrt[3]{243}$$

**Surd law 2**

$$\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}, \ (a, b \in \mathbb{Z}^+, \ n \in \mathbb{N})$$

For example:

$$\frac{\sqrt[3]{243}}{\sqrt[3]{9}} = \sqrt[3]{\frac{243}{9}}$$

$$= \sqrt[3]{27}$$

$$= 3$$

And:

$$\frac{\sqrt{32}}{\sqrt{50}} = \sqrt{\frac{32}{50}}$$

$$= \sqrt{\frac{16}{25}}$$

$$= \frac{4}{5}$$
Surd law 3

\[ (\sqrt[n]{a})^m = \sqrt[n]{a^m}, \ (a \in \mathbb{Z}^+, \ m, n \in \mathbb{N}) \]

For example:

\[ (\sqrt[9]{4})^4 = (\sqrt[9]{4^4}) \]
\[ = 9^{\frac{4}{4}} \]
\[ = 9^1 \]
\[ = 9 \]

And:

\[ (\sqrt{39})^2 = \sqrt{39^2} \]
\[ = 39 \]

Surd law 4

\[ \sqrt[\text{m}]{\sqrt[n]{a}} = \sqrt[\text{mn}]{a}, \ (a \in \mathbb{Z}^+, \ m, n \in \mathbb{N}) \]

For example:

\[ \sqrt[4]{\sqrt[5]{6561}} = \sqrt[4 \times 5]{6561} \]
\[ = (6561)^{\frac{1}{5}} \]
\[ = (3^8)^{\frac{1}{5}} \]
\[ = 3 \]

And:

\[ \sqrt[3]{\sqrt[3]{729}} = \sqrt[3 \times 3]{729} \]
\[ = (729)^{\frac{1}{3 \times 3}} \]
\[ = (3^6)^{\frac{1}{3 \times 3}} \]
\[ = 3 \]

In the following section, you will learn how to apply the surd laws when solving problems that include surds.
Calculations with surds

In this section, we'll explain how to work with surds. Specifically, we'll discuss the three operations that you'll encounter most often when working with surds. These operations enable you to:

- simplify surds using prime factors;
- make use of the order of a surd; and
- rationalise a denominator containing surds.

Simplify surds using prime factors

Surds can be simplified using prime factors. To do so, reduce each value in the expression to prime factors, and then look for ways to group these factors together. These groupings often enable you to simplify the overall mathematical expression.

Let’s work through a few examples to see how to simplify expressions containing surds.

**EXAMPLE**

Simplify without using a calculator:

1. \(\sqrt[3]{216}\)

2. \(\frac{\sqrt{98} \times \sqrt{50}}{\sqrt{32}}\)

3. \(\sqrt{72} + \sqrt{32} - \sqrt{200}\)

**Answers**

1. \(\sqrt[3]{216} = \sqrt[3]{2 \times 2 \times 2 \times 3 \times 3 \times 3}\) (prime factors)

   \[= \sqrt[3]{(2)^3} \times (3)^3\] (group as cubes)

   \[= 2 \times 3\]

   \[= 6\]

2. \(\frac{\sqrt{98} \times \sqrt{50}}{\sqrt{32}} = \frac{\sqrt{49 \times 2} \times \sqrt{25 \times 2}}{\sqrt{16 \times 2}}\)

   \[= \frac{7 \times \sqrt{2} \times 5 \times \sqrt{2}}{4 \times 4 \times \sqrt{2}}\] (prime factors)

   \[= \frac{\sqrt{7^2 \times 2} \times \sqrt{5^2 \times 2}}{\sqrt{4^2 \times 4}}\]

   \[= \frac{\sqrt{7^2} \times \sqrt{2} \times \sqrt{5^2} \times \sqrt{2}}{\sqrt{4^2} \times \sqrt{2}}\]
\[
\frac{7\sqrt{2} \times 5\sqrt{2}}{4\sqrt{2}} = \frac{7 \times 5 \times \sqrt{2}}{4} = \frac{35\sqrt{2}}{4}
\]

(simplified surds)

(cancel equal factors)

(simplify)

\[= \frac{35}{4} \sqrt{2}\]

3. \[\sqrt{72} + \sqrt{32} - \sqrt{200} = \sqrt{36 \times 2 + 16 \times 2 - 100 \times 2} = \sqrt{6^2 \times 2 + 4^2 \times 2 - 10^2 \times 2} = \sqrt{6^2 \sqrt{2} + 4^2 \sqrt{2} - 10^2 \sqrt{2}} = 6\sqrt{2} + 4\sqrt{2} - 10\sqrt{2} \quad \text{(or, taking out a common factor of } \sqrt{2} \text{ we get } \sqrt{2}(6 + 4 - 10) = \sqrt{2} (0) = 0) \]

The order of a surd

Remember that a surd is an irrational number in the form \(\sqrt[n]{ma}\).

In this form, \(n\) is the order of the surd. If two surds have the same order, then they must have the same value for \(n\).

For example, \(\sqrt[4]{23}\), \(\sqrt[4]{29}\) and \(\sqrt[4]{71}\) are surds of the same order. Specifically, these surds are of order 4.

One important use of the order of a surd is that it enables you to arrange them in either ascending order or descending order of value.

Let’s look at the next example shows.

**EXAMPLE**

Arrange the following three surds in descending order:

\[\sqrt{2}; \sqrt[4]{5}; \sqrt[3]{4}\]

**Answers**

To arrange the surds in descending order, we need to:

- first rewrite them in exponential form;
- then rewrite each exponent where the denominators are the same; and
- then convert each exponential form back to surd form.
Let’s express each one in exponential form:

- \( \sqrt{2} = (2)^{\frac{1}{2}} \);
- \( \sqrt[4]{5} = (5)^{\frac{1}{4}} \); and
- \( \sqrt[3]{4} = (4)^{\frac{1}{3}} \).

Next, we need to change the fractions so that they all have the same denominator.

The lowest common multiple (LCM) of the denominators of the indices 2, 4 and 3 is 12.

Therefore, we need to rewrite all the indices with denominator 12, and then rewrite the exponential expression back to surd form:

\[
\sqrt{2} = (2)^{\frac{1}{2}} \\
= (2)^{\frac{6}{12}} \\
= \frac{\sqrt{2^6}}{\sqrt{12}} \\
= \frac{\sqrt{64}}{\sqrt{12}} \\
= \frac{8}{2\sqrt{3}} \\
= \frac{4\sqrt{3}}{3}
\]

\[
\sqrt[4]{5} = (5)^{\frac{1}{4}} \\
= (5)^{\frac{3}{12}} \\
= \frac{\sqrt[3]{5^3}}{\sqrt[4]{125}} \\
= \frac{\sqrt[3]{125}}{\sqrt[4]{125}} \\
= \frac{5}{\sqrt[4]{5}}
\]

\[
\sqrt[3]{4} = (4)^{\frac{1}{3}} \\
= (4)^{\frac{4}{12}} \\
= \frac{\sqrt[4]{4^4}}{\sqrt[3]{125}} \\
= \frac{\sqrt[4]{256}}{\sqrt[3]{125}} \\
= \frac{16}{\sqrt[3]{125}}
\]

Now, since the order of each of the surds is the same, we can directly compare the values represented by \( m \) in the general form of a surd.

Therefore:

\[
\frac{16}{\sqrt[3]{125}} > \frac{\sqrt[4]{125}}{5} > \frac{\sqrt{2}}{3}
\]

And, if we express the surds in their original form, we have:

\[
\sqrt[3]{4} > \sqrt[4]{5} > \sqrt{2}.
\]
Rationalise a denominator containing surds

Since a surd is an irrational number, any number expressions that include a surd are also irrational.

Therefore, expressions such as \( \frac{2}{\sqrt{3}} \) and \( \frac{5}{\sqrt{2} - 1} \) have irrational denominators.

To rationalise a denominator of a surd, it means the denominator contains no surds.

There are essentially two situations that you may find when trying to rationalise a denominator, namely:

- the denominator contains one term only; or
- the denominator contains more than one term.

If the denominator only contains a single term, then it can be rationalised by multiplying the expression by:

\[
\frac{\text{denominator}}{\text{denominator}}
\]

However, if there is more than one term in the denominator, then the expression needs to be multiplied by:

\[
\frac{\text{sum or difference of two terms of the denominator}}{\text{sum or difference of two terms of the denominator}}
\]

What this mean is if the denominator contains a sum of two terms, then the expression needs to be multiplied by:

\[
\frac{\text{difference of two terms of the denominator}}{\text{difference of two terms of the denominator}}
\]

Conversely, if the denominator contains a difference of two terms, then the expression needs to be multiplied by:

\[
\frac{\text{sum of two terms of the denominator}}{\text{sum of two terms of the denominator}}
\]

For example, \( \sqrt{3} - 1 \) is an example of the difference of two terms.

Then the expression must be multiplied by \( \frac{\sqrt{3} + 1}{\sqrt{3} + 1} \) to rationalise the denominator if \( \sqrt{3} - 1 \) is the denominator.

Similarly, \( 5 + \sqrt{2} \) is an example of the sum of two terms. Then the expression must be multiplied by \( \frac{5 - \sqrt{2}}{5 - \sqrt{2}} \) to rationalise the denominator if \( 5 + \sqrt{2} \) is the denominator.

The numerator remains irrational. Only the denominator must be rational.

Let's look at following examples.
EXAMPLE

Rationalise the denominator in each of the following expressions:

1. \( \frac{7}{\sqrt{5}} \)

2. \( \frac{2}{\sqrt{11} - 9} \)

Answers

1. \( \frac{7}{\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}} = \frac{7\sqrt{5}}{(\sqrt{5})^2} \)
   
   \( = \frac{7\sqrt{5}}{5} \)

2. \( \frac{2}{\sqrt{11} - 9} \times \frac{\sqrt{11} + 9}{\sqrt{11} + 9} = \frac{2\sqrt{11} + 18}{(\sqrt{11})^2 + 9\sqrt{11} - 9\sqrt{11} - 81} \)
   
   \( = \frac{2\sqrt{11} + 18}{11 - 81} \)
   
   \( = \frac{2\sqrt{11} + 18}{-70} \)

Now work carefully through the following activity to practise what you have learnt about working with surds.

Activity 5

1. Simplify without using a calculator:
   
   (a) \( \frac{\sqrt{45}}{\sqrt{180}} \)
   
   (b) \( \frac{\sqrt{48} - \sqrt{27}}{\sqrt{12}} \)

2. Arrange the following surds in ascending order (without using a calculator):
   
   \( \sqrt{5} ; \sqrt{8} ; \frac{10}{\sqrt{25}} \)

3. Rationalise the denominator of the following expressions:
   
   (a) \( \frac{2 - \sqrt{2}}{\sqrt{8}} \)
   
   (b) \( \frac{1}{\sqrt{7} - 2} \)
Answers

1. (a) \( \frac{\sqrt{45}}{\sqrt{180}} = \frac{\sqrt{9 \times 5}}{\sqrt{36 \times 5}} \)
   
   = \frac{3\sqrt{5}}{6\sqrt{5}}
   
   = \frac{3}{6}
   
   = \frac{1}{2}

(b) \( \frac{\sqrt{48} - \sqrt{27}}{\sqrt{12}} = \frac{\sqrt{16 \times 3} - \sqrt{9 \times 3}}{\sqrt{4 \times 3}} \)

   = \frac{4\sqrt{3} - 3\sqrt{3}}{2\sqrt{3}}

   = \frac{\sqrt{3}}{2\sqrt{3}}

   = \frac{1}{2}

2. Rewrite all surd form to exponential form:

\( \sqrt[4]{5} \); \( \sqrt[12]{125} \); \( \sqrt[2]{25} \)

\( \sqrt[4]{5} = (5)^{\frac{1}{4}} \); \( \sqrt[12]{125} = \sqrt[3]{5^{3}} \); \( \sqrt[2]{25} = \sqrt[2]{5^{2}} \)

\( = (5)^{\frac{1}{4}} \); \( = (5)^{\frac{3}{2}} \); \( = (5)^{\frac{2}{3}} \)

\( = (5)^{\frac{3}{12}} \); \( = (5)^{\frac{18}{12}} \); \( = (5)^{\frac{8}{12}} \)

\( = 12\sqrt[3]{5}^{3} \); \( = 12\sqrt[18]{5}^{18} \); \( = 12\sqrt[8]{5}^{8} \)

Now you can compare the values to each other, and order them as follows:

\( 12\sqrt[3]{5}^{3} < 12\sqrt[18]{5}^{18} < 12\sqrt[8]{5}^{8} \)

And expressed in the original form of each surd, you have:

\( \sqrt[4]{5} < \sqrt[2]{25} < \sqrt{125} \)
3. (a) There is only one term in the denominator.

\[
\frac{2 - \sqrt{2}}{\sqrt{8}} \times \frac{\sqrt{8}}{\sqrt{8}} = \frac{2\sqrt{8} - \sqrt{2} \times \sqrt{8}}{8}
\]

\[
= \frac{2\sqrt{4 \times 2} - \sqrt{16}}{8}
\]

\[
= \frac{2 \times 2 \times \sqrt{2} - 4}{8}
\]

\[
= \frac{4\sqrt{2} - 4}{8}
\]

\[
= \frac{4(\sqrt{2} - 1)}{8} \text{ (take out a common factor)}
\]

\[
= \frac{\sqrt{2} - 1}{2}
\]

(b) There are two terms in the denominator.

\[
\frac{1}{\sqrt{7} - 2} \times \frac{\sqrt{7} + 2}{\sqrt{7} + 2} = \frac{\sqrt{7} + 2}{7 - 4}
\]

\[
= \frac{\sqrt{7} + 2}{3}
\]

**Summary**

In Lesson 3, you learnt that a surd is an irrational number of the form \(\sqrt[|n|]{a^m}\).

There are four surd laws that we derived directly from the index laws. These laws are:

- **Surd law 1:** \(\sqrt[n]{a} \times \sqrt[n]{b} = \sqrt[n]{ab}\), \((a, b \in Z^+, n \in N)\)
- **Surd law 2:** \(\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}\), \((a, b \in Z^+, n \in N)\)
- **Surd law 3:** \((\sqrt[n]{a})^m = \sqrt[n]{a^m}\), \((a \in Z^+, m, n \in N)\)
- **Surd law 4:** \(m\sqrt[n]{a} = \sqrt[n]{a^m}\), \((a \in Z^+, m, n \in N)\)

Finally, you learnt about some of the ways that you are able to work with surds. Specifically, you learnt how to:

- simplify surds using prime factors;
- make use of the order of a surd; and
- rationalise a denominator containing surds.
Self-assessment Questions 3

Test your knowledge of this lesson by completing the self-assessment questions below. When you answer the questions, don’t look at the suggested answers that we give straight away. Look at them only after you’ve written your answers down, and then compare your answers with our answers. Remember, no calculators must be used.

1. Simplify the expression \((\sqrt{50} - \sqrt{32})^2\).

2. Simplify and rationalise the denominator of the expression \(\frac{\sqrt{8} \times \sqrt{2}}{\sqrt{6}}\).

3. Simplify the expression \(\frac{\sqrt{48} - \sqrt{27}}{\sqrt{147}}\).

4. Simplify the expression \(\sqrt{72} + \sqrt{128} - \sqrt{392}\).

5. Solve for \(x\) in the following equation: \(5^{x-2} = 5\sqrt{5}\).

6. Simplify the expression \(\frac{\sqrt{343} - \sqrt{7}}{\sqrt{7}}\).

7. Simplify the expression \(\frac{3\sqrt{8} + \sqrt{3} - \sqrt{75}}{3\sqrt{2} - \sqrt{12}}\).

8. Solve for \(x\) in the following equation: \(1 - 2x + \sqrt{5x - 1} = 0\).

Suggested answers to Self-assessment Questions 3

1. \((\sqrt{50} - \sqrt{32})^2 = (\sqrt{25 \times 2} - \sqrt{16 \times 2})^2\)
   
   \[= (5\sqrt{2} - 4\sqrt{2})^2\]
   
   \[= (\sqrt{2})^2\]
   
   \[= 2\]

2. \(\frac{\sqrt{8} \times \sqrt{2}}{\sqrt{6}} = \frac{\sqrt{4} \times 2 \times \sqrt{2}}{\sqrt{2} \times 3}\)
   
   \[= \frac{2\sqrt{2} \times \sqrt{2}}{\sqrt{2} \times \sqrt{3}}\]
   
   \[= \frac{2\sqrt{2} \times \sqrt{2}}{\sqrt{3} \times \sqrt{3}}\]
\[
\frac{\sqrt{48} - \sqrt{27}}{\sqrt{147}} = \frac{\sqrt{16 \times 3} - \sqrt{9 \times 3}}{\sqrt{49 \times 3}} \\
= \frac{4\sqrt{3} - 3\sqrt{3}}{7\sqrt{3}} \\
= \frac{\sqrt{3}}{7\sqrt{3}} \\
= \frac{1}{7}
\]

4. \[
\sqrt{72} + \sqrt{128} - \sqrt{392} = \sqrt{36 \times 2} + \sqrt{64 \times 2} - \sqrt{196 \times 2} \\
= 6\sqrt{2} + 8\sqrt{2} - 14\sqrt{2} \\
= 0
\]

5. \[
5^{x - 2} = 5\sqrt{5} \\
= 5^1 \times 5^\frac{1}{2} \\
= 5^{\frac{3}{2}}
\]
Therefore:
\[
x - 2 = \frac{3}{2} \\
x = 3 \frac{1}{2}
\]

6. \[
\frac{\sqrt{343} - \sqrt{7}}{\sqrt{7}} = \frac{\sqrt{49 \times 7} - \sqrt{7}}{\sqrt{7}} = 7\sqrt{7} - \sqrt{7} \\
= \frac{\sqrt{7}(7 - 1)}{\sqrt{7}} \\
= \sqrt{7}(6) \\
= 6
\]
7. \[ \frac{3\sqrt{8} + \sqrt{3} - \sqrt{75}}{3\sqrt{2} - \sqrt{12}} = \frac{3 \times 2\sqrt{2} + \sqrt{3} - 5\sqrt{3}}{3\sqrt{2} - 2\sqrt{3}} \]
\[ = \frac{6\sqrt{2} - 4\sqrt{3}}{3\sqrt{2} - 2\sqrt{3}} \]
\[ = \frac{2(3\sqrt{2} - 2\sqrt{3})}{(3\sqrt{2} - 2\sqrt{3})} \]
\[ = 2 \]

8. \[ 1 - 2x + \sqrt{5x-1} = 0 \]
\[ \sqrt{5x-1} = 2x - 1 \quad \text{(isolate the surd)} \]
\[ 5x - 1 = (2x - 1)^2 \quad \text{(square both sides)} \]
\[ 5x - 1 = 4x^2 - 4x + 1 \]
\[ -4x^2 + 9x - 2 = 0 \]
\[ 4x^2 - 9x + 2 = 0 \]
\[ (4x - 1)(x - 2) = 0 \]

Therefore, there are two possible solutions to the equation:
\[ x = \frac{1}{4} \]
or
\[ x = 2 \]

Test each of the two possible answers for validity.

For \( x = \frac{1}{4} \), you have:
\[ \text{LHS: } 1 - 2(\frac{1}{4}) + \sqrt{5(\frac{1}{4}) - 1} = 1 - \frac{1}{2} + \sqrt{\frac{5}{4} - 1} \]
\[ = \frac{1}{2} + \sqrt{\frac{1}{4}} = \frac{1}{2} + \frac{1}{2} \]
\[ = 1 \neq 0 \]
\[ \neq \text{RHS} \]

Therefore, \( x = \frac{1}{4} \) is not a valid solution.

For \( x = 2 \), you have:
\[ \text{LHS: } 1 - 2(2) + \sqrt{5(2) - 1} = 1 - 4 + \sqrt{10 - 1} \]
\[ = -3 + \sqrt{9} \]
\[ = -3 + 3 \]
\[ = 0 \]
\[ = \text{RHS} \]

Therefore, \( x = 2 \) is a valid solution.
Check your competence

Now that you have worked through this lesson, please check that you can perform the tasks below:

- I understand that not all numbers are real.
- I can simplify expressions using the laws of exponents for rational exponents.
- I can add, subtract, multiply and divide simple surds.
- I can demonstrate an understanding of error margins.
- I can solve non-routine, unseen problems.

The next lesson

If you are sure that you understand the contents covered in this lesson, and have achieved your Assessment Standards, start Lesson 4.
LESSON 4: LOGARITHMS: BASIC PRINCIPLES AND APPLICATIONS

Learning Outcomes for Lesson 4

After you have worked through Lesson 4, you should be able to do the following:

<table>
<thead>
<tr>
<th>LO</th>
<th>Assessment Standard</th>
</tr>
</thead>
<tbody>
<tr>
<td>LO 1</td>
<td>AS 11.1.1 Understand that not all numbers are real. (This requires the recognition, but not the study of non-real numbers)</td>
</tr>
<tr>
<td></td>
<td>AS 11.1.2 (a) Simplify expressions using the laws of exponents for rational exponents.</td>
</tr>
<tr>
<td></td>
<td>(b) Add, subtract, multiply and divide simple surds.</td>
</tr>
<tr>
<td></td>
<td>(c) Demonstrate an understanding of error margins.</td>
</tr>
<tr>
<td></td>
<td>AS 11.1.6 Solve non-routine, unseen problems.</td>
</tr>
<tr>
<td></td>
<td>AS 12.1.2 Demonstrate an understanding of the definition of a logarithm and any laws needed to solve real-life problems (for example, growth and decay).</td>
</tr>
<tr>
<td></td>
<td>AS 12.1.6 Solve non-routine, unseen problems.</td>
</tr>
</tbody>
</table>

Remember to use the checkbox once you’ve completed this lesson. You’ll find it right at the end of the lesson.

Introduction

In Lesson 4, you’ll learn about logarithms. We’ll begin by defining a logarithm, and you’ll see how logarithms are closely related to numbers expressed in exponential form.

As with surds, there are laws that govern logarithms. These laws follow directly from the laws of indices, just as was the case with the surd laws.

In the last two sections of the lesson, we'll discuss how to apply the logarithm laws. Specifically, you’ll learn how to:

- simplify expressions without using a calculator; and
- solve exponential equations by using a calculator if the bases cannot be made the same.

So, the main sections of this lesson are as follows:

- The definition of a logarithm
- The logarithmic laws
- Simplifying expressions containing logarithms
- Solving exponential equations using logarithms
The definition of a logarithm

The word 'logarithm' is a little cumbersome, and so you'll often find it abbreviated to the word 'log'. We'll use the same convention in this unit as well, and will use the two words interchangeably.

The formal definition of a logarithm is as follows:

\[ \log_a x = b \text{ if and only if } a^b = x, \text{ for } a > 0, \ a \neq 1 \text{ and } x > 0. \]

Let's look more closely at this definition. Logarithms use the same terminology as exponential expressions. The definition of a logarithm says that the logarithm of \( x \), if the base is \( a \), is the index or exponent to which \( a \) must be raised to produce \( x \).

For example, the expression \( \log_2 16 = 4 \) means that \( \log \) of 16 is the index to which 2 must be raised to produce 16. In other words, 2 needs to be raised to the power of 4 in order to produce the value 16. Therefore, the exponential form of the same expression is \( 2^4 = 16 \).

Table 1 shows a few more examples of numbers expressed in both the logarithmic form and the exponential form.

<table>
<thead>
<tr>
<th>Logarithmic form</th>
<th>Exponential form</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \log_7 1 = 0 )</td>
<td>( 7^0 = 1 )</td>
</tr>
<tr>
<td>( \log_2 8 = -3 )</td>
<td>( \left(\frac{1}{2}\right)^{-3} = 8 )</td>
</tr>
<tr>
<td>( \log_5(5^x) = x )</td>
<td>( 5^x = 5^x )</td>
</tr>
</tbody>
</table>

A negative value for either \( a \) or \( x \) in the definition of a log is not defined.

For example, consider the expression \( \log_5 (-8) \). There is no power of 5 (or any positive number) that will be negative. Therefore, \( \log_5 (-8) \) does not exist.

In general, the expression \( \log_a x \) is only defined for values \( a \) and \( x \) greater than 0, and for \( a \) not equal to 1.

Before the general availability of calculators, logs were a very important aid to doing complicated calculations. Originally, mathematicians always took the base of a log to be 10. These days, we still refer to logs with base 10 as common logs.

If an expression does not show the base of a log, then assume that the base is 10. For example, \( \log 7 \) is the same as \( \log_{10} 7 \).

Although logarithms do not play the same role in calculations as they used to, they are still of fundamental importance in mathematical calculations. No training in basic mathematics will be comprehensive if it does not include the theory of logarithms.

In the next section, we’ll look at the laws that govern logarithms.
The logarithmic laws

As mentioned in the introduction to this lesson, the log laws follow directly from the laws of indices. There are four log laws.

Log law 1

\[ \log_a(cd) = \log_a c + \log_a d \]

For example:

\[ \log_3 10 = \log_3 5 + \log_3 2 \]

However, note that:

\[ \log_a (c+d) \neq \log_a c + \log_a d \]

And:

\[ \log_a c \times \log_a d \neq \log_a (cd) \]

Log law 2

\[ \log_a \left( \frac{c}{d} \right) = \log_a c - \log_a d \]

For example:

\[ \log_7 50 = \log_7 100 - \log_7 2 \]

However, note that:

\[ \log_a (c-d) \neq \log_a c - \log_a d \]

And:

\[ \frac{\log_a c}{\log_a d} \neq \log_a \left( \frac{c}{d} \right) \]

Log law 3

\[ \log_a(c^t) = t \log_a c \]

For example:

\[ \log_3 3^2 = 2 \log_3 3^2 \]

However, note that:

\[ (\log_a c)^t \neq t \log_a c \]
Log law 4

\[ \log_b c = \frac{\log_a c}{\log_a b} \]

For example, \( \log_3 15 = \frac{\log_{10} 15}{\log_{10} 3} \)

Now let’s see how to apply the logarithmic laws to help simplify mathematical expressions.

**Simplifying expressions containing logarithms**

The log laws enable us to simplify number expressions that contain logarithms, or to rewrite them in another form. The following examples illustrate how to apply the log laws when simplifying expressions.

---

**EXAMPLE**

Simplify the following expressions by using the log laws, and without using a calculator:

1. \( 2 \log 3 + \log 5 - 3 \log 2 \)
2. \( 2 \log 5 - \log 4 - \log 2 \)
3. \( \log_2 (32) \)
4. \( \frac{\log 9 + \log 4}{\log 6} \)
5. \( \frac{\log 5 - \log 3}{\log 25 - \log 9} \)
6. \( 2 \log_3 27 + 2 \log_6 6 - \frac{1}{2} \log_7 49 \)

**Answers**

1. \( 2 \log 3 + \log 5 - 3 \log 2 = \log 3^2 + \log 5 - \log 2^3 \) (log law 3)
   \[ = \log 9 + \log 5 - \log 8 \]
   \[ = \log \frac{9 \times 5}{8} \] (log laws 1 and 2)
   \[ = \log \frac{45}{8} \]

2. \( 2 \log 5 - \log 4 - \log 2 = \log 5^2 - (\log 4 + \log 2) \)
   \[ = \log 25 - \log (4 \times 2) \] (log law 1)
   \[ = \log \frac{25}{8} \] (log law 2)
3. \[\log_2 (32)^{-1} = \log_2 \left(2^5\right)^{-1}\]
   \[= \log_2 (2)^{-5}\]
   \[= -5 \log_2 2\]
   \[= -5 \times 1\]
   \[= -5\]  
   (Log law 3)
   (Note: \(\log_a a = 1\), since \(a^1 = a\))

4. \[\frac{\log 9 + \log 4}{\log 6} = \frac{\log (9 \times 4)}{\log 6}\]  
   (log law 1)
   \[= \frac{\log 36}{\log 6}\]
   \[= \frac{\log 6^2}{\log 6}\]
   \[= 2 \frac{\log 6}{\log 6}\]  
   (log law 3)
   \[= 2\]  
   (cancel equal factors)

5. There are two ways to solve this problem.

   Method 1:
   \[\frac{\log 5 - \log 3}{\log 25 - \log 9} = \frac{\log 5 - \log 3}{\log 5^2 - \log 3^2}\]
   \[= \frac{\log 5 - \log 3}{2 \log 5 - 2 \log 3}\]
   \[= \frac{\log 5 - \log 3}{2(\log 5 - \log 3)}\]
   \[= \frac{1}{2}\]

   Method 2:
   \[\frac{\log 5 - \log 3}{\log 25 - \log 9} = \frac{\log \left(\frac{5}{3}\right)}{\log \left(\frac{25}{9}\right)}\]
   \[= \frac{\log \left(\frac{5}{3}\right)}{\log \left(\frac{5}{3}\right)^2}\]
   \[= \frac{\log \left(\frac{5}{3}\right)}{2 \log \left(\frac{5}{3}\right)}\]
   \[= \frac{1}{2}\]
6. There are two ways to solve this problem.

Method 1:
\[
\log_3 27 + 2 \log_6 6 - \frac{1}{2} \log_7 49 = \frac{\log 27}{\log 3} + 2 \frac{\log 6}{\log 6} - \frac{1}{2} \frac{\log 49}{\log 7}
\]
\[
= \frac{\log 3^3}{\log 3} + 2 - \frac{1}{2} \frac{\log 7^2}{\log 7}
\]
\[
= 3 \frac{\log 3}{\log 3} + 2 - \frac{1}{2} \frac{2 \log 7}{\log 7}
\]
\[
= 3 + 2 - 1 = 4
\]

Method 2:
\[
\log_3 27 = \log_3 3^3 = 3 \log_3 3 = 3 \times 1 = 3
\]
\[
\log_6 6 = 1
\]
\[
\log_7 49 = \log_7 7^2 = 2 \log_7 7 = 2 \times 1 = 2
\]

Therefore:
\[
\log_3 27 + 2 \log_6 6 - \frac{1}{2} \log_7 49 = 3 + 2 \times 1 - \frac{1}{2} \times 2
\]
\[
= 3 + 2 - 1 = 4
\]

Now work carefully through the following activity.

**Activity 6**

1. Without a calculator, find the value of the following expressions:
   (a) \( \log_2 0,125 \)
   (b) \( \log_5 \left( \frac{1}{125} \right) \)
   (c) \( \log 1 \)
   (d) \( \log 0 \)

2. Rewrite the following expressions in exponential form:
   (a) \( \log_2 64 = 6 \)
   (b) \( \log_3 \left( \frac{1}{81} \right) = -4 \)
3. Rewrite the following expressions in log form:

(a) \( 3^{-2} = \frac{1}{9} \)

(b) \( 10^3 = 1\,000 \)

(c) \( \sqrt{25} = 5 \)

4. Simplify without a calculator:

(a) \( \frac{\log 125 - \log 64}{\log 25 - \log 16} \)

(b) \( \frac{2 \log 3 - \log 27}{\left(\log \frac{1}{9}\right)(\log 100)} \)

(c) \( \log_4 8 + \log_8 4 \)

5. If \( \log 2 = 0,31 \) and \( \log 3 = 0,48 \), calculate the value of \( \log 0,75 \).

**Answers**

1. (a) \( \log_2 0,125 = \log_2 \left(\frac{1}{8}\right) \)

   \( = \log_2 (2)^{-3} \)

   \( = -3 \log_2 2 \)

   \( = -3 \times 1 \)

   \( = -3 \)

(b) \( \log_5 \left(\frac{1}{125}\right) = \log_5 (5)^{-3} \)

   \( = -3 \log_5 5 \)

   \( = -3 \times 1 \)

   \( = -3 \)

(c) \( \log 1 = 0 \) \hspace{2cm} (since \( 10^0 = 1 \))

(d) \( \log 0 \) is undefined

2. (a) \( 2^6 = 64 \)

(b) \( (3)^{-4} = \frac{1}{81} \)
3. (a) \( \log_3 \left( \frac{1}{9} \right) = -2 \)

(b) \( \log 1\,000 = 3 \)

(c) \( \log_{25} 5 = \frac{1}{2} \) \( \left( 25^{\frac{1}{2}} = 5 \right) \)

4. (a) \( \frac{\log 5^3 - \log 2^6}{\log 5^2 - \log 2^4} = \frac{3 \log 5 - 6 \log 2}{2 \log 5 - 4 \log 2} \)

\[ = \frac{3 (\log 5 - 2 \log 2)}{2 (\log 5 - 2 \log 2)} \]

\[ = \frac{3}{2} \]

(b) \( \frac{2 \log 3 - \log 3^3}{(\log(3)^{-2}) \times 2} = \frac{2 \log 3 - 3 \log 3}{(-2 \log 3) \times 2} \) (remember, \( \log 100 = 2 \))

\[ = \frac{-\log 3}{-4 \log 3} \]

\[ = \frac{1}{4} \]

(c) \( \frac{\log 8}{\log 4} + \frac{\log 4}{\log 8} = \frac{3 \log 2}{2 \log 2} + \frac{2 \log 2}{3 \log 2} \)

\[ = \frac{3}{2} + \frac{2}{3} \]

\[ = \frac{13}{6} \]

5. \( \log 0,75 = \log \frac{75}{100} \)

\[ = \log \frac{3}{4} \]

\[ = \log 3 - \log 4 \]

\[ = \log 3 - 2 \log 2 \]

\[ = 0,48 - 2(0,30) \) (substitute given values)

\[ = -0,12 \]

In the last section of this lesson, we'll explain how to solve exponential equations using logs.
Solving exponential equations using logarithms

In Lesson 1 you learned that, to solve an exponential equation, you need to write both sides with the same base, and then equate the indices on each side.

For example:

\[5^x = 25\]
\[= 5^2\]

Therefore:

\[x = 2\]

However, what happens when it is impossible to make the two bases the same? The answer is to use the log form of each expression, rather than the exponential form.

When using logs to solve an equation, you'll often have to use a calculator to work out the correct answer. Therefore, even though we can theoretically use any base for the log expressions, we tend to choose base 10. The reason is that most calculators can only calculate log values using base 10.

Let's look at an example.

**EXAMPLE**

Solve for \(x\) in the following equation: \(5^{x-1} = 17\). Express your answer to four decimal places.

**Answer**

\[
\log 5^{x-1} = \log 17
\]
\[(x - 1) \log 5 = \log 17 \quad \text{(log law 3)}\]
\[x - 1 = \frac{\log 17}{\log 5} \quad \text{(divide both sides by log 5)}\]
\[x = 1 + \frac{\log 17}{\log 5}\]
\[x = 2.7604 \quad \text{(to four decimal places)}\]

Be careful when calculating the value of \(\frac{\log 17}{\log 5}\).

In particular, note the following:

\[
\log 17 \over \log 5 \neq \log 17 \over 5
\]

And:

\[
\log 17 \over \log 5 \neq \log 17 - \log 5
\]

We calculated the final answer using a calculator, and rounded it off to four decimal places.
Now work carefully through the following activity to practise what you have learnt about solving exponential equations using logs.

**Activity 7**
Solve for \( x \) in the following equations. Express your answer to four decimal places.

1. \( 4 \times 3^x = 17 \)
2. \( 3^x = 6^{2+x} \)
3. \( 2^x \times 5^{x+1} = 86 \)

**Answers**

1. \( 3^x = \frac{17}{4} \)
   
   \[
x \log 3 = \log \left( \frac{17}{4} \right)
   \]
   
   \[
x = \frac{\log \left( \frac{17}{4} \right)}{\log 3}
   \]
   
   (divide both sides by \( \log 3 \))
   
   \[
x = 1.3170 \quad \text{(answer to four decimal places)}
   \]

2. \( 3^x = 6^{2+x} \)
   
   \[
x \log 3 = (2 + x) \log 6
   \]
   
   \[
x \log 3 = 2 \log 6 + x \log 6
   \]
   
   \[
x \log 3 - x \log 6 = 2 \log 6
   \]
   
   \[
x (\log 3 - \log 6) = 2 \log 6
   \]
   
   \[
x = \frac{2 \log 6}{\log 3 - \log 6}
   \]
   
   \[
x = -5.1699 \quad \text{(to four decimal places)}
   \]

3. \( 2^x \times 5^{x+1} = 86 \)
   
   \[
\log(2^x \times 5^{x+1}) = 86
   \]
   
   \[
\log 2^x + \log 5^{x+1} = \log 86 \quad \text{(log law 1)}
   \]
   
   \[
x \log 2 + (x + 1) \log 5 = \log 86
   \]
   
   \[
x \log 2 + x \log 5 + \log 5 = \log 86
   \]
   
   \[
x (\log 2 + \log 5) = \log 86 - \log 5
   \]
   
   \[
x = \frac{\log 86 - \log 5}{\log 2 + \log 5}
   \]
   
   \[
x = 1.2355 \quad \text{(to four decimal places)}
   \]
Summary

Lesson 4 covered logarithms (logs).

You learnt that logarithms and exponents are closely related. The logarithmic form of an expression is simply a different way of writing that same expression in exponential form.

The definition of a log states that \( \log_a x = b \) if and only if \( a^b = x \), for values of \( a > 0 \), \( a \neq 1 \) and \( x > 0 \). In other words, the logarithm of \( x \), if the base is \( a \), is the index or exponent to which \( a \) must be raised to produce \( x \).

You also learnt about the four important log laws that govern what you can and can’t do with log calculations.

Finally, in the last two sections of the lesson, you learnt how to apply the logarithm laws, by learning how to:

- simplify expressions without using a calculator; and
- solve exponential equations by using a calculator, if the bases cannot be made the same.

Self-assessment Questions 4

Test your knowledge of this lesson by completing the self-assessment questions below. When you answer the questions, don’t look at the suggested answers that we give straight away. Look at them only after you’ve written your answers down, and then compare your answers with our answers.

1. Rewrite the following expressions in exponential form:
   (a) \( \log_3 x = a \)
   (b) \( \log_2 \left( \frac{1}{8} \right) = -3 \)

2. Rewrite the following expressions in logarithmic form:
   (a) \( (5)^{-2} = \frac{1}{25} \)
   (b) \( 0.125 = (8)^{-1} \)

3. Without using a calculator, find the value of:
   (a) \( \log_4 \sqrt[3]{16} \)
   (b) \( \log_9 \left( \frac{1}{27} \right) \)
   (c) \( \frac{\log_2 4 + \log_2 3}{2 + \log_2 36} \)
4. Simplify without using a calculator:

(a) \( \log_3 \sqrt[4]{27} + \log_3 \sqrt{3} \)

(b) \( \frac{\log 81 - \log 9}{\log 81 - \log \left(\frac{1}{3}\right)} \)

(c) \( \log_9 27 - \log_{27} 9 \)

5. Solve for \( x \) in the following equations. Express your answer to three decimal places:

(a) \( 5 \times 3^{2x} = 16 \)

(b) \( 2^x + 15 \times 2^{-x} = 8 \)

Suggested answers to Self-assessment Questions 4

1. (a) \( 3^a = x \)

(b) \( (2)^3 = \frac{1}{8} \)

2. (a) \( \log_5 \left(\frac{1}{25}\right) = -2 \)

(b) \( \log_9 0.125 = -1 \)

3. (a) \( \log_4 \sqrt[3]{16} = \log_4 (16)^{\frac{1}{3}} \)

\[ = \frac{1}{3} \log_4 4^2 \]

\[ = \frac{2}{3} \log_4 4 \]

\[ = \frac{2}{3} \]

(b) \( \log_9 \left(\frac{1}{27}\right) = \log_9 3^{-3} \)

\[ = -3 \times \log_9 3 \]

\[ = -3 \times \frac{\log 3}{\log 9} \]

\[ = -3 \times \frac{\log 3}{\log 3^2} \]
\[
= -3 \times \frac{\log 3}{2 \log 3}
= -\frac{3}{2}
\]

(c) \[
\frac{\log_2 4 + \log_2 3}{2 + \log_2 36} = \frac{2 \log_2 2 + \log_2 3}{2 + \log_2 (2^2 \times 3^2)}
= \frac{2 \log_2 2 + \log_2 3}{2 + 2 \log_2 2 + 2 \log_2 3}
= \frac{2 + \log_2 3}{4 + 2 \log_2 3}
= \frac{2 + \log_2 3}{2(2 + \log_2 3)}
= \frac{1}{2}
\]

4. (a) \[
\log_3 \sqrt[4]{27} + \log_3 \sqrt{3} = \log_3 (27)^{\frac{1}{4}} + \log_3 (3)^{\frac{1}{2}}
= \frac{3}{4} \log_3 3 + \frac{1}{2} \log_3 3
= \frac{3}{4} + \frac{1}{2}
= 1 \frac{1}{4}
\]

(b) \[
\frac{\log 81 - \log 9}{\log 81 - \log \left(\frac{1}{3}\right)} = \frac{\log_9 9 - \log 9}{2 \log 9 + \log 9}
= \frac{\log_9 9}{3 \log 9}
= \frac{1}{3}
\]

(c) \[
\log_9 27 - \log_9 9 = \frac{3 \log 3}{2 \log 3} - \frac{2 \log 3}{3 \log 3}
= \frac{3}{2} - \frac{2}{3}
= \frac{5}{6}
\]
Alternatively \( \log_3 27 - \log_{27} 9 = \frac{\log 27}{\log 9} - \frac{\log 9}{\log 27} \)
\[
= \frac{\log 3^3}{\log 3^2} - \frac{\log 3^2}{\log 3^3}
= \frac{3 \log 3}{2 \log 3} - \frac{2 \log 3}{3 \log 3}
= \frac{3}{2} - \frac{2}{3}
= \frac{5}{6}
\]

5. (a) \( 5 \times 3^{2x} = 16 \)
\[
3^{2x} = \frac{16}{5}
\]
\[
2x \log 3 = \log \left( \frac{16}{5} \right)
\]
\[
x = \frac{\log \left( \frac{16}{5} \right)}{2 \log 3}
\]
\[
= 0.529
\]

(b) \( 2^x + 15 \times 2^{-x} = 8 \)
\[
2^{2x} - 8 \times 2^x + 15 = 0 \quad \text{(multiply both sides by } 2^x)\]
\[
(2^x - 3)(2^x - 5) = 0
\]
Therefore, there are two possible solutions to the equation, namely:
\( 2^x = 3 \)
And:
\( 2^x = 5 \)

Solve for \( x \) in the first possible solution:
\[
2^x = 3
\]
\[
x \log 2 = \log 3 \quad \text{(take the log of both sides)}
\]
\[
x = \frac{\log 3}{\log 2}
\]
\[
= 1.585
\]
Solve for \( x \) in the second possible solution:

\[
2^x = 5
\]

\[
x \log 2 = \log 5 \quad \text{(take the log of both sides)}
\]

\[
x = \frac{\log 5}{\log 2}
\]

\[
= 2.322
\]

### Check your competence

Now that you have worked through this lesson, please check that you can perform the tasks below:

- I understand that not all numbers are real.
- I can simplify expressions using the laws of exponents for rational exponents.
- I can add, subtract, multiply and divide simple surds.
- I can demonstrate an understanding of error margins.
- I can demonstrate an understanding of the definition of a logarithm and any laws needed to solve real-life problems (for example, growth and decay).
- I can solve non-routine, unseen problems.

### The next lesson

If you are sure that you understand the contents covered in this lesson, and have achieved your Assessment Standards, start Lesson 5.
Learning Outcomes for Lesson 5

After you have worked through Lesson 5, you should be able to do the following:

<table>
<thead>
<tr>
<th>LO</th>
<th>Assessment Standard</th>
</tr>
</thead>
<tbody>
<tr>
<td>LO 1</td>
<td>AS 11.1.2 (a) Simplify expressions using the laws of exponents for rational exponents.</td>
</tr>
<tr>
<td></td>
<td>(b) Add, subtract, multiply and divide simple surds.</td>
</tr>
<tr>
<td></td>
<td>(c) Demonstrate an understanding of error margins.</td>
</tr>
<tr>
<td></td>
<td>AS 11.1.3 Investigate number patterns (including, but not limited to those where there is a constant second difference between consecutive terms in a number pattern, and the general term is therefore quadratic) and hence:</td>
</tr>
<tr>
<td></td>
<td>(a) make conjectures and generalisations</td>
</tr>
<tr>
<td></td>
<td>provide explanations and justifications, and attempt to prove conjectures.</td>
</tr>
<tr>
<td></td>
<td>AS 11.1.6 Solve non-routine, unseen problems.</td>
</tr>
<tr>
<td></td>
<td>AS 12.1.3 (a) Identify and solve problems involving number patterns, including, but not limited to arithmetic and geometric sequences and series. Correctly interpret recursive formulae (for example, $T_{n+1} = T_n + T_{n-1}$).</td>
</tr>
<tr>
<td></td>
<td>AS 12.1.6 Solve non-routine, unseen problems.</td>
</tr>
</tbody>
</table>

Remember to check your competence at the end of this lesson.

Introduction

The focus of Lesson 5 is on arithmetic sequences and series. However, we begin the lesson by looking at number patterns in more detail. In particular, we’ll look at:

- linear relations;
- quadratic relations; and
- exponential relations.

In each case, we’ll also explain how to determine the type of relation represented by a particular number pattern. You’ll learn how to use the method of calculating the first and second difference for the number pattern, and to use this information to determine the type of relation. Next, we’ll explain what a mathematical sequence is in general terms. Here, we’ll also explain the difference between a sequence and a series.
When we look at arithmetic sequences in more detail, you will learn how to determine the components of an arithmetic sequence, including:

- the common difference of the sequence; and
- the general term of the sequence.

Finally, we’ll explain how to solve problems that involve arithmetic sequences.

**Number patterns**

A number relation shows a relationship between two variables, usually $x$ and $y$. In other words, we can describe a particular number pattern by means of a mathematical formula. In this section, we’ll examine three particular number relations, namely:

- linear relations;
- quadratic relations; and
- exponential relations.

**Linear relations**

The following formula represents a linear relation:

$$y = mx + c$$

The graph of this relation is a straight line. Figure 1 shows an example of a straight-line graph that is described by the equation $y = -3x + 2$.

![Figure 1](image)
Figure 1 shows two points on the graph that have been plotted.

Table 2 shows these and other possible points on the graph.

<p>| TABLE 2 |</p>
<table>
<thead>
<tr>
<th>A LINEAR RELATIONSHIP</th>
</tr>
</thead>
<tbody>
<tr>
<td>x</td>
</tr>
<tr>
<td>-2</td>
</tr>
<tr>
<td>-1</td>
</tr>
<tr>
<td>0</td>
</tr>
<tr>
<td>1</td>
</tr>
<tr>
<td>2</td>
</tr>
</tbody>
</table>

Is it possible to recognise the relationship as a linear? The answer is yes. The method you use in this case is to calculate what we call the first difference.

To calculate the first difference, work out the difference between successive y values in the set of numbers. If the value of the first difference is the same, then the relationship is a linear one.

Table 3 shows the same information contained in Table 2, but also includes the first difference for each line.

Note how the value remains constant for each successive line. The calculation in brackets in each line of the 'First difference' column shows how to calculate that particular value.

<p>| TABLE 3 |</p>
<table>
<thead>
<tr>
<th>A LINEAR RELATIONSHIP</th>
</tr>
</thead>
<tbody>
<tr>
<td>x</td>
</tr>
<tr>
<td>-2</td>
</tr>
<tr>
<td>-1</td>
</tr>
<tr>
<td>0</td>
</tr>
<tr>
<td>1</td>
</tr>
</tbody>
</table>

If the first difference between any two consecutive y values is the same (or constant), then the relationship between x and y is linear.

Now we know that the relationship shown in Table 3 is a linear one. However, it is possible to work out the equation of the relation from the table.

Remember, the general equation for a linear relationship is $y = mx + c$. Therefore, to derive the equation for the linear relation, we need to calculate the numerical values for m and c.
The following method is one way to determine these values:

- **Step 1:** The first difference, by calculating the difference between consecutive y values, is the numerical value for \( m \). In the example shown in Table 3, this value is -3. Therefore, the equation becomes \( y = -3x + c \).

- **Step 2:** To calculate the value for \( c \), substitute any coordinate \((x;y)\) from the table and solve for \( c \). Therefore, if we use, for example, \((-2;8)\) and substitute it in the equation \( y = -3x + c \), we obtain the following:

  \[
  \therefore 8 = -3(-2) + c \\
  \therefore 8 = 6 + c \\
  \therefore 8 - 6 = c \\
  \therefore 2 = c
  \]

Therefore, the equation of the graph is \( y = -3x + 2 \).

**Quadratic relations**

The following formula represents a quadratic relation:

\[
\sum y = ax^2 + bx + c
\]

We call the graph of a quadratic relation a parabola. Figure 2 shows an example of a parabola.

![Graph of a parabola](image)

**Fig. 2**  
A parabola

*Table 4* shows the information for a quadratic relation. We have included a fourth column, which contains the value of the second difference. We calculate the first difference in exactly the same way as we did for a linear relation.
We calculate the second difference by working out the difference between successive values in the 'First difference' column.

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
<th>First difference</th>
<th>Second difference</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>-4</td>
<td>-5 (-4 – 1)</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>-13</td>
<td>-9 (-13 – (-4))</td>
<td>-4 (-9 – (-5))</td>
</tr>
<tr>
<td>3</td>
<td>-26</td>
<td>-13 (-26 – (-13)</td>
<td>-4 (-13 – (-9))</td>
</tr>
<tr>
<td>4</td>
<td>-43</td>
<td>-17 (-43 – (-26)</td>
<td>-4 (17 – (-13))</td>
</tr>
</tbody>
</table>

As you can see from Table 4, the values in the 'First difference' column are not the same. Therefore, this is not a linear relation. However, the values in the 'Second difference' column are the same.

The rule is that, if the values in the 'Second difference' column are the same, then the relation is a quadratic relation.

If the second difference between any two consecutive y values is the same or constant, then the relationship between x and y is quadratic.

Remember that the standard form for a quadratic relation is \( y = ax^2 + bx + c \).

Once again, as with linear equations, it’s possible to work out the equation for the quadratic equation by using the information in the table of first and second differences.

Follow the steps below to work out the equation for the quadratic equation:

- **Step 1:** To calculate the value of \( a \), we take the numerical value of the second difference and divide by 2, that is:

\[
a = \frac{\text{second difference}}{2}
\]

From our example above, the second difference is -4. Therefore:

\[
a = \frac{-4}{2} = -2
\]

- **Step 2:** To find the values of \( b \) and \( c \), we substitute any two pairs of coordinates and use simultaneous equations to solve \( b \) and \( c \).

Therefore, if we use: (1; -4) and (2; -13), we get the following:

\[
y = ax^2 + bx + c
\]

\[
y = -2x^2 + bx + c
\]

\[-4 = 2(1)^2 + b(1) + c\] (substitute in the values from the point (1; -4)
Next, we substitute \((2; –13)\) into the equation \(y = ax^2 + bx + c\). Remember that \(a = -2\).

Therefore:

\[
\begin{align*}
y &= ax^2 + bx + c \\
-13 &= -2(2)^2 + b(2) + c \\
-13 &= -8 + 2b + c \\
-5 &= 2b + c
\end{align*}
\]

Now we solve \(-2 = b + c\) and \(-5 = 2b + c\) simultaneously.

Therefore, from \(-2 = b + c\), when we make \(c\) the subject, we get:

\[c = -b - 2\]

Substituting \(c = -b - 2\) into \(-5 = 2b + c\), we get the following:

\[
\begin{align*}
-5 &= 2b + (-b - 2) \\
-5 &= 2b - b - 2 \\
\therefore -3 &= b
\end{align*}
\]

Therefore, \(b = -3\).

To find the value of \(c\), we substitute \(b = -3\) into either \(-2 = b + c\) or \(-5 = 2b + c\). We will use \(-2 = b + c\).

Therefore we get

\[
\begin{align*}
-2 &= -3 + c \\
1 &= c
\end{align*}
\]

Therefore, the equation of the graph is \(y = -2x^2 - 3x + 1\). Note that we could have chosen any point from the table of data to calculate the value of \(b\).

**Exponential relations**

The following formula represents an exponential relation:

\[
\sum y = Ab^x
\]

*Figure 3* shows an example of an exponential curve.
We can use the first and second difference information to identify an exponential relation, just as we did for the linear and quadratic relations. However, the values in the first and second difference columns are not constant as they were for the first two relations.

Table 5 shows the data for an exponential relation. (Note that this is not the exponential relation that we illustrated in Figure 3.)

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y$</th>
<th>First difference</th>
<th>Second difference</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>7</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>11</td>
<td>4 (11 − 7)</td>
<td>8 (12 − 4)</td>
</tr>
<tr>
<td>2</td>
<td>23</td>
<td>12 (23 − 11)</td>
<td>8 (12 − 4)</td>
</tr>
<tr>
<td>3</td>
<td>59</td>
<td>36 (59 − 23)</td>
<td>24 (36 − 12)</td>
</tr>
<tr>
<td>4</td>
<td>167</td>
<td>108 (167 − 59)</td>
<td>72 (108 − 36)</td>
</tr>
</tbody>
</table>

As you can see from Table 5, neither the first nor the second values are constant. Can you see another relationship between the numbers?

If we calculate the ratio of successive values in the first or second difference columns, then we'll find that these ratios are actually constant.

Table 6 shows the results of the ratio calculations for the 'First difference' column.
### TABLE 6
AN EXPONENTIAL RELATIONSHIP INCLUDING RATIO CALCULATIONS

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y$</th>
<th>First difference</th>
<th>Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>7</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>11</td>
<td>4 (11 − 7)</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>23</td>
<td>12 (23 − 11)</td>
<td>$3 \left(\frac{12}{4}\right)$</td>
</tr>
<tr>
<td>3</td>
<td>59</td>
<td>36 (59 − 23)</td>
<td>$3 \left(\frac{36}{12}\right)$</td>
</tr>
<tr>
<td>4</td>
<td>167</td>
<td>108 (167 − 59)</td>
<td>$3 \left(\frac{108}{36}\right)$</td>
</tr>
</tbody>
</table>

We can conclude that, if the ratio of successive values in the first difference column is constant, then the relation is an exponential relation. Note that the ratio of the successive values in the second difference column is also constant. However, a constant ratio in the first difference column is enough to identify the relation as exponential.

If the **ratio** of successive values in the first difference column is constant, then the relation is an **exponential** relation.

Remember that the general form of the equation for the exponential equation is $y = Ab^x$. Just as before, we can use the information in the data table to work out the equation for the relation by following these steps:

- **Step 1**: The value of $b$ is the constant ratio that we calculated in Table 6. In this case, the ratio is equal to 3. Therefore, we now have the equation $y = A3^x$.

- **Step 2**: Substitute one of the data pairs into the equation from Step 1, and calculate the value of $A$. If you have it available, then the line containing $x = 0$ is the simplest to use. This gives us:

  $$ y = A3^x $$

  $$ 7 = A \times 3^0 $$

  $$ 7 = A \times 1 $$

  $$ 7 = A $$

Therefore, the equation for the exponential graph represented by the data in Table 6 is $y = 7 \times 3^x$.

In the next section, we’ll discuss **sequences** in more detail.
Sequences

Let’s start by looking at the definition of a sequence.

A sequence is a list of numbers that appear in a specified order. We call these numbers terms, and separate them using commas or semicolons.

The numbers in a sequence usually follow a rule. In other words, the terms in the sequence are not just random numbers, but follow a pattern. This pattern is the rule that defines the numbers in the sequence.

A sequence can be finite, which means that there are a limited number of terms in the sequence, or it can be infinite. The following example shows a few different sequences. Try to identify the rule that defines each sequence.

**EXAMPLE**

What is the rule that defines each of the following sequences?

1. 0; 3; 6; 9; 12; 15; 18; 21
2. -3; -7; -11; -15; -19; . . .
3. -2; -4; -8; -16; -32; -64
4. log 4; log 20; log 100; log 500; log 2 500; log 12 500
5. $3^\frac{1}{3}; 3^\frac{2}{3}; 3; 3^2; 3^4; 3^8; 3^{16}; 3^{32}; . . .$
6. 4; 9; 25; 49; 121; 169; . . .
7. $\frac{1}{3}; \frac{1}{9}; \frac{1}{27}; \frac{1}{81}; . . .$

**Answers**

The rule for each of the sequences is as follows:

1. Add 3 to each successive value.
2. Subtract 4 from each successive value or add –4 to each successive value.
3. Multiply each successive value by 2.
4. We can rewrite this sequence as follows:
   
   log 4; log (4 × 5); log (4 × 25); log (4 × 125); . . .

   Or:

   log 4; log 4 + log 5; log 4 + 2 log 5; log 4 + 3 log 5; . . .

   Therefore the rule is that we add (log 5) to each successive term.
5. Square each term to produce the next term.
6. The sequence is the squared value of the list of prime numbers. In other words, we can rewrite the sequence as:

\[ 2^2; 3^2; 5^2; 7^2; 11^2; 13^2; \ldots \]

7. Multiply each successive value by \( \frac{1}{3} \).

We indicate the first term of a sequence by \( t_1 \), the second term by \( t_2 \), and so on. As soon as we know the rule according to which each position in the sequence is occupied, we can write down the \( n \)th term, \( t_n \).

Sometimes we need to determine the sum of the terms of a sequence. In other words, we add up all the values in the sequence. We indicate the sum of the first two terms by \( S_2 \), the sum of the first three terms by \( S_3 \), and in general, the sum of the first \( n \) terms by \( S_n \).

Technically speaking the word sequence has a different meaning than series, although they are related. A series originates when we add the terms of a sequence together. In other words, the terms in a series are separated by plus signs, rather than semicolons.

There are two general types of sequences and series, namely:

- arithmetic sequences and series;
- geometric sequences and series.

In the remainder of this lesson, we’ll examine arithmetic sequences and series in more detail. In the next lesson, you’ll learn all about geometric sequences and series.

**Arithmetic sequences and series**

In this section, we’re going to discuss arithmetic sequences and series. As you’ll see, arithmetic sequences and series are linear relations.

In the next lesson, you’ll discover that geometric sequences and series are exponential relations.

Let’s begin by formally defining an arithmetic sequence.

If a sequence \( t_1; t_2; t_3; \ldots \) has the property that:

\[ t_2 = t_1 + d \]
\[ t_3 = t_2 + d \]
\[ t_4 = t_3 + d \]

And:

\[ t_n = t_{n-1} + d \]

for a number \( d \) (which can be positive, negative or zero), then such a sequence is an arithmetic sequence (AS).
We call the number $d$ that we add to each term to produce the next term the common difference ($d$).

According to the above definition:

$$t_2 - t_1 = t_3 - t_2 = t_4 - t_3 = \ldots = t_n - t_{n-1} = d$$

The number $d$ must remain constant over the entire length of the sequence.

**The general term of an arithmetic sequence**

Now suppose that we have an AS with $t_1 = a$, and a common difference of $d$. Then:

$$t_2 = a + d$$

$$t_3 = t_2 + d = a + 2d$$

$$t_4 = t_3 + d = a + 3d$$

From this, we can deduce the following formula for the general term in an arithmetic sequence:

$$t_n = a + (n - 1)d$$

where $t_n$ is the $n$th term, or general term.

**The sum of the terms in an arithmetic series**

We can also deduce a formula for the sum of the first $n$ terms of an arithmetic series.

Suppose once again that we have an AS with $t_1 = a$, and a common difference of $d$. Then:

$$S_n = t_1 + t_2 + \ldots + t_{n-1} + t_n$$

$$= a + (a + d) + \ldots + (a + (n - 2)d) + (a + (n - 1)d)$$

Now let's rewrite the equation for the sum of the series, but this time writing the terms in the reverse order. In other words, we start with the last term $(a + (n - 1)d)$, and finish with the first term. When we add all the terms in the second equation, then the sum will still be $S_n$.

Therefore, in the second equation, we have:

$$S_n = (a + (n - 1)d) + (a + (n - 2)d) + \ldots + (a + d) + a$$

Now, let's add the two equations together. The left-hand side is easy, because we just need to add $S_n + S_n$, which equals $2S_n$.

Adding the two sets of values on the right-hand side is a bit more complicated.
However, if we group the terms so that we add the first term of each equation together, then add the second terms together, and so on, then the result is as follows:

\[
\begin{align*}
\quad & a + a + d + \ldots + a + (n-2)d + a + (n-1)d \\
\quad & a + (n-1)d + a + (n-2)d + \ldots + a + d + a \\
\quad & 2a + (n-1)d + 2a + (n-1)d + \ldots + 2a + (n-1)d + 2a + (n-1)d
\end{align*}
\]

Therefore, the result of adding the two equations together is:

\[2S_n = (2a + (n-1)d) + (2a + (n-1)d) + \ldots + (2a + (n-1)d) + (2a + (n-1)d)\]

On the right-hand side of the resulting equation, we see something remarkable. There are \(n\) terms between the plus signs, and each term is equal to \(2a + (n-1)d\). Therefore, we can rewrite this result as follows:

\[2S_n = n(2a + (n-1)d)\]

Now, remember that the value that we actually want is the sum of the terms in the series. In other words, we want the value of \(S_n\). We can obtain this by dividing both sides of the equation by 2.

Therefore, the general formula for the sum of the terms in an arithmetic series is:

\[\sum S_n = \frac{n}{2} (2a + (n-1)d)\]

However, we’re still not finished! There is one other form that we can use to express the formula for the sum of an arithmetic series. Remember that the final term, (or general term) of the series is given by \(a + (n-1)d\).

Therefore, if we represent this final term as \(l\), then we can rewrite the formula as:

\[\sum S_n = \frac{n}{2} (a + l)\]

where \(l\) is the final term in the series.

Let’s see how to use all this information to solve problems that include an arithmetic sequence or series.

**EXAMPLE**

An AS has a third term of 14, and a seventh term of 26. Calculate the sum of the first seven terms.

**Answer**

We know that:

\[t_3 = a + 2d\]

\[t_7 = a + 6d\]
Therefore:

\[ a + 2d = 14 \quad \text{(from } t_3) \]

\[ a = 14 - 2d \]

And:

\[ a + 6d = 26 \quad \text{(from } t_7) \]

If we substitute the value of \( a \) from the first equation into the second equation, then we can calculate the value of \( d \), as follows:

\[ 14 - 2d + 6d = 26 \]

\[ 4d = 12 \]

\[ d = 3 \]

Now, we can calculate the value of \( a \), as follows:

\[ a = 14 - 2d \]

\[ = 14 - 6 \]

\[ = 8 \]

Now:

\[ S_n = \frac{n}{2} (a + l) \]

Therefore:

\[ S_7 = \frac{7}{2} (t_1 + t_7) \]

\[ = \frac{7}{2} (8 + 26) \]

\[ = 7 \times 17 \]

\[ = 119 \]

The sum of the first seven terms is 119.

In the next example, we begin by knowing what the sum of part of an arithmetic series is, but we don’t know how many terms form part of that sum.
EXAMPLE

How many terms of the infinite series $2 + 7 + 12 + \ldots$ must be added to give a sum of 297?

Answer

We know that:

- the first term in the series ($a$) is 2; and
- the common difference ($d$) is 5.

We also know that $S_n = 297$.

Therefore:

$$S_n = \frac{n}{2} (2a + (n - 1)d)$$

$$297 = \frac{n}{2} (2 \times 2 + (n - 1) \times 5)$$

$$= \frac{n}{2} (4 + 5n - 5)$$

$$= \frac{n}{2} (5n - 1)$$

$$594 = n (5n - 1)$$

$$= 5n^2 - n$$

$$0 = 5n^2 - n - 594$$

Now we can factorise the quadratic equation, and solve for $n$.

$$n = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{1 \pm \sqrt{(-1)^2 - 4(5)(-594)}}{2(5)}$$

$$= \frac{1 \pm \sqrt{11881}}{10}$$

$$= \frac{1 \pm 109}{10}$$

$$= 11 \text{ or } -10.8$$

Now we know that $n$ can only be a whole number, because it refers to the number of terms in the series. Therefore, $n$ must be equal to 11. In other words, eleven terms of the series must be added to reach a total of 297.
In the final example of this lesson, we once again know the formula for the sum of \( n \) terms in a series, but now we need to prove that the series is an arithmetic series.

**EXAMPLE**

A series is such that the sum of the first \( n \) terms is \( 2n^2 - 3n \).

1. Prove that the series is an arithmetic series.
2. Find the value of the 18th term.

**Answer**

1. We know that the general formula for the sum of the series is \( S_n = 2n^2 - 3n \).

   Therefore:
   
   \[
   S_1 = 2 - 3 = -1
   \]
   \[
   S_2 = 8 - 6 = 2
   \]
   \[
   S_3 = 18 - 9 = 9
   \]

   We also know that:
   
   \[
   t_1 = S_1 = -1
   \]
   \[
   t_2 = S_2 - S_1 = 2 - (-1) = 3
   \]
   \[
   t_3 = S_3 - S_2 = 9 - 2 = 7
   \]

   Therefore the sequence is \(-1; 3; 7; \ldots\), which is clearly an AS with:

   \[
   a = -1
   \]
   \[
   d = 4
   \]

2. The 18th term is:

   \[
   t_{18} = a + 17d = -1 + 17 \times 4 = 67
   \]
Now work carefully through the following activity to practise what you have learnt about number patterns and arithmetic sequences and series.

### Activity 8

1. (a) State the nature of the relationship for each of the following sets of data.

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<td>20</td>
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2. Determine which of the following sequences are arithmetic sequences (AS). For each AS, give the value of \( a \) and \( d \).

   (a) 5; 8; 11; 14; . . .
   (b) 5, 8; 5, 2; 4, 6; 4; . . .
   (c) \( \frac{1}{2}; 1; 2; 4; . . . \)
   (d) \( x + y; x; x - y; x - 2y; . . . \)
   (e) \( \log 2; \log 4; \log 8; \log 16; . . . \)

3. (a) Find the twelfth term of the sequence 7; 4; 1; . . .
   (b) Calculate the sum of the first 20 terms of the sequence in (a).

4. In an AS, the thirteenth term is \(-21\), and the 27th term is \(-49\). Find the first three terms of the AS.

5. How many terms of the sequence 3; 5; 7; 9; . . . must be added to produce a sum of 224?

6. The eighth term of an AS is \(-1\), and the sum of the first ten terms is 40. Find the number of terms in this AS for which the sum is 0.

7. During a freefall, a parachutist falls 5 m during the first second, 15 m during the second second, and 25 m during the third second. If he continues in this way:

   (a) How far will he fall during the tenth second?
   (b) What is the total distance that he falls during the first 10 seconds?
Answer

1. (a) i. The second difference is a constant value of 2. Therefore, the relationship is a quadratic relationship.

ii. The second difference is a constant value of 4. Therefore, the relationship is a quadratic relationship.

ii. The first difference is a constant value of 4. Therefore, the relationship is a linear relationship.

ii. There is a constant ratio between successive first difference values of 2. Therefore, the relationship is an exponential relationship.

2. (a) This is an AS, with \( a = 5 \) and \( d = 3 \).

(b) This is an AS with \( a = 5,8 \) and \( d = -0,6 \).

(c) This is not an AS. The rule in this case is to multiply each value by 2 to arrive at the next value.

(d) \[ t_2 - t_1 = x - (x + y) \]
   \[ = -y \]
   \[ t_3 - t_2 = x - y - x \]
   \[ = -y \]
   \[ t_4 - t_3 = (x - 2y) - (x - y) \]
   \[ = -y \]

Therefore, this is an AS, with \( a = x + y \) and \( d = -y \).

(e) \[ t_2 - t_1 = \log 4 - \log 2 \]
   \[ = 2 \log 2 - \log 2 \]
   \[ = \log 2 \]

\[ t_3 - t_2 = \log 8 - \log 4 \]
   \[ = 3 \log 2 - 2 \log 2 \]
   \[ = \log 2 \]

\[ t_4 - t_3 = \log 16 - \log 8 \]
   \[ = 4 \log 2 - 3 \log 2 \]
   \[ = \log 2 \]

Therefore, this is an AS, with \( a = \log 2 \) and \( d = \log 2 \).
3. (a) The sequence is an AS, with $a = 7$ and $d = -3$. Therefore the twelfth term is:

\[
t_{12} = a + 11d = 7 + 11(-3) = -26
\]

(b) The sum of the first 20 terms in the series is:

\[
S_{20} = \frac{20}{2} (2(7) + 19(-3)) = -430
\]

4. The sequence is an AS, so you need to calculate the values of $a$ and $d$. Once you have those two values, you can work out the first three terms in the sequence.

\[
t_{13} = a + 12d = -21
\]

Therefore, $a = -12d - 21$.

\[
t_{27} = a + 26d = -49
\]

If you substitute the value for $a$ from the first equation into the second equation, then you have:

\[
-12d - 21 + 26d = -49
\]

\[
14d = -28
\]

\[
d = -2
\]

Now you can calculate $a$ from the first equation:

\[
a = -12d - 21 = 24 - 21 = 3
\]

Therefore, the first three terms of the series are 3, 1 and -1.

5. The sequence is an AS, with $a = 3$ and $d = 2$. You also know that $S_n = 224$. You need to solve for $n$ in the following formula:

\[
\frac{n}{2} (2(3) + (n - 1)2) = 224 \\
\frac{n(6 + 2n - 2)}{2} = 448 \quad \text{(Multiply both sides by 2)}
\]

\[
4n + 2n^2 = 448
\]

\[
n^2 + 2n - 224 = 0
\]

\[
(n - 14)(n + 16) = 0
\]

Therefore, the two possible solutions to the equation are $n = -16$ and $n = 14$.

Since $n$ must be greater than 0, $n = 14$. Therefore, the sum of the first 14 terms is equal to 224.
6. You know the following information:

\[ a + 7d = -1 \]  \hspace{1cm} \text{(from the term } t_8 \text{)}

\[ a = -1 - 7d \]

And:

\[ \frac{10}{2} (2a + 9d) = 40 \]  \hspace{1cm} \text{(the sum of the first 10 terms)}

\[ 10(2a + 9d) = 80 \]

\[ 2a + 9d = 8 \]

Now you can substitute the value of \( a \) from the first equation into the second equation, and calculate the value of \( d \):

\[ 2(-1 - 7d) + 9d = 8 \]
\[ -2 - 14d + 9d = 8 \]
\[ -5d = 10 \]
\[ d = -2 \]

If you substitute \( d \) back into the first equation, you get:

\[ 2a + 9(-2) = 8 \]
\[ 2a = 8 + 18 \]
\[ 2a = 26 \]
\[ a = 13 \]

The question asks you to work out how many terms you need to add together to get a result of 0. To work this out, you need to solve for \( n \) in the following equation:

\[ \frac{n}{2} \left( 2(13) + (n - 1)(-2) \right) = 0 \]
\[ n (26 - 2n + 2) = 0 \]
\[ n (28 - 2n) = 0 \]

Therefore, the two possible solutions to the equation are:

\[ n = 0 \]

And:

\[ 28 - 2n = 0 \]
\[ 28 = 2n \]
\[ n = 14 \]

Since \( n \) must be greater than 0, the only valid solution is \( n = 14 \). In other words, the sum of the first 14 terms in the series is 0.
7. (a) The distance that the parachutist falls in each second is described by the arithmetic sequence:

5; 15; 25; . . .

The first value of the sequence \( (a) \) is 5, and the common difference \( (d) \) is equal to 10. Therefore, the tenth value in the sequence is:

\[
t_{10} = a + (n - 1)d
\]
\[
= 5 + 9(10)
\]
\[
= 95
\]

Therefore, the parachutist falls 95m during the tenth second.

(b) The total distance that the parachutist falls is the sum of the first 10 terms of the series. Therefore, the total distance is:

\[
S_{10} = \frac{10}{2} (2(5) + 9(10))
\]
\[
= 500
\]

Therefore, the parachutist will have fallen 500m after 10 seconds.

Summary

Lesson 5 covered two main areas, namely:

- number patterns; and
- arithmetic sequences and series.

First, we discussed number patterns, and the number relations that represent certain patterns. You learnt how to examine a set of data values, and to work out what sort of relation the data represented. You learnt how to look for certain relationships in the first and second difference for the data, and to use this information to decide on:

- the type of relation; and
- the mathematical equation that described the relation.

The three types of number relations that you learnt about in this lesson were:

- linear relations;
- quadratic relations; and
- exponential relations.

The second area covered in the lesson was that of arithmetic sequences and series. Here you learnt that a sequence is a list of values, and that a rule defines each value in the list. You also learnt that a sequence becomes a series when you look to add the values in the sequence together.

If the rule that defines the sequence states that you need to add a constant value to each term in order to produce the value of the next term, then the sequence is an arithmetic sequence. As a result, an arithmetic sequence is a linear relation.
If the rule that defines the sequence requires that you multiply a constant number to each term to produce the next term, then the sequence is a geometric sequence. A geometric sequence is an exponential relation.

In this lesson, you learnt that it’s possible to define an arithmetic sequence by specifying the value of the first term \( a \), and the common difference between each term \( d \). These two values allow you to define the general term, or \( n \)th term of the sequence.

Similarly, you learnt how to calculate the general formula to add up the values in an arithmetic series.

Finally, we demonstrated how to solve various types of problems that include an arithmetic sequence or series.

In the next lesson, you’ll learn more about geometric sequences and series.

**Self-assessment Questions 5**

Test your knowledge of this lesson by completing the self-assessment questions below. When you answer the questions, don’t look at the suggested answers that we give straight away. Look at them only after you’ve written your answers down, and then compare your answers with our answers.

1. Write down the first three terms of an AS that has an \( n \)th term defined by:
   \[ t_n = -2n + 5 \]
2. The first three terms of an AS are \( 3p - 4 \), \( 4p - 3 \), and \( 7p - 6 \).
   (a) Solve for \( p \), and so determine the values of the first three terms of the sequence.
3. Determine the number of terms in each of the following arithmetic sequences:
   (a) \( 18; 25; 32; \ldots; 697 \)
   (b) \( -4; 1; 6; \ldots; 66 \)
4. The 6th term of an AS is 5, and the sum of the first 10 terms is 65. Determine the sum of the first 20 terms.
5. A runner training for a marathon runs 30 km in his first week, and increases his training distance by 5 km every week.
   (a) In which week would he cover 85 km?
   (b) How many weeks must he run to complete 1 080 km?
6. The sum of the first ten terms of an AS is 145, and the sum of the fourth and ninth terms is five times the third term. Write down the first three terms of the AS.
7. Which term of the AS \( 11p; 13p; 15p; \ldots \) will be 243\( p \)?
8. The sum of \( n \) terms of an arithmetic sequence is \( n(n + 2) \). Find the value of the tenth term.
9. The sum of eleven terms of the series \[ \frac{\log_2 x}{2} + \frac{\log_3 x}{3} + \frac{\log_4 x^3}{4} + \ldots \] is 242.

   (a) Determine the value of \( x \).
   (b) Find the value of the tenth term.

10. Find the value of \( \frac{199 + 197 + 195 + \ldots + 101}{99 + 97 + 95 + \ldots + 1} \).

### Suggested answers to Self-assessment Questions 5

1. You know that \( t_n = -2n + 5 \). Therefore:

   \[
   t_1 = -2(1) + 5 = 3 \\
   t_2 = -2(2) + 5 = 1 \\
   t_3 = -2(3) + 5 = -1
   \]

   Therefore, the first three terms of the sequence are 3, 1 and \(-1\).

2. (a) Since the sequence is an arithmetic sequence, you know that the difference between successive terms is constant. In other words:

   \[
   t_2 - t_1 = t_3 - t_2
   \]

   Therefore:

   \[
   4p - 3 - (3p - 4) = 7p - 6 - (4p - 3) \\
   4p - 3 - 3p + 4 = 7p - 6 - 4p - 3 \\
   4p - 3p - 7p + 4p = -6 + 3 + 3 - 4 \\
   -2p = -4 \\
   p = 2
   \]
Therefore, the first three terms of the sequence are:

\[ t_1 = 3p - 4 \]
\[ = 3(2) - 4 \]
\[ = 2 \]
\[ t_2 = 4p - 3 \]
\[ = 4(2) - 3 \]
\[ = 5 \]
\[ t_3 = 7p - 6 \]
\[ = 7(2) - 6 \]
\[ = 8 \]

3. (a) The first term of the sequence is 18 (the value of \( a \)). The common difference for the sequence is 7 (the value of \( d \)). Therefore, the formula for the last term is:

\[ t_n = a + (n - 1)d \]
\[ = 18 + (n - 1)7 \]
\[ = 11 + 7n \]

Now you know that the last term in the sequence is 697. Therefore:

\[ 697 = 11 + 7n \]
\[ 7n = 686 \]
\[ n = 98 \]

There are 98 terms in the sequence.

(b) The first term of the sequence is \(-4\) (a). The common difference for the sequence is 5 (d). Therefore, the formula for the last term is:

\[ t_n = a + (n - 1)d \]
\[ = -4 + (n - 1)5 \]
\[ = -9 + 5n \]
You know that the last term in the sequence is 66. Therefore:

\[ 66 = -9 + 5n \]

\[ 5n = 75 \]

\[ n = 15 \]

There are 15 terms in the sequence.

4. The formula for the sum of the first 20 terms is:

\[ S_{20} = \frac{n}{2} (2a + (n - 1)d) \]

\[ = 10(2a + 19d) \]

\[ = 20a + 190d \]

As you can see, you need to know the values of \( a \) and \( d \) to be able to work out the sum of the first 20 terms. You can work out these values by using the other information given to you in the question.

You know that the 6th term of the sequence is 5. Therefore:

\[ t_6 = a + (n - 1)d \]

\[ 5 = a + 5d \]

\[ a = 5 - 5d \]

You also know that the sum of the first 10 terms is 65. Therefore:

\[ S_{10} = \frac{n}{2} (2a + (n - 1)d) \]

\[ 65 = 5(2a + 9d) \]

\[ 65 = 10a + 45d \]

Now you can substitute the value of \( a \) from the first equation into the second equation, which gives you:

\[ 65 = 10(5 - 5d) + 45d \]

\[ = 50 - 50d + 45d \]

\[ 5d = 50 - 65 \]

\[ = -15 \]

\[ d = -3 \]
Now you can substitute this value for $d$ back into the first equation, and work out the value of $a$, as follows:

$$a = 5 - 5d$$

$$= 5 - 5(-3)$$

$$= 20$$

Now you can calculate the sum of the first 20 terms:

$$S_{20} = 20a + 190d$$

$$= 20(20) + 190(-3)$$

$$= 400 - 570$$

$$= -170$$

5. (a) The first term of the sequence is 30 ($a$). The common difference for the sequence is 5 ($d$). Therefore, you need to solve for $n$ in the general formula for the sequence. Since $t_n = 85$:

$$85 = a + (n - 1)d$$

$$= 30 + (n - 1)5$$

$$= 25 + 5n$$

$$60 = 5n$$

$$n = 12$$

Therefore, the runner would cover 85km in the 12th week.

(b) Here you need to solve for $n$ in the general formula for the sum of $n$ terms. Once again, $a$ is equal to 30, and $d$ is equal to 5. You also know that $S_n = 1080$.

Therefore:

$$1080 = \frac{n}{2} (2a + (n - 1)d)$$

$$= \frac{n}{2} (2(30) + (n - 1)5)$$

$$= \frac{n}{2} (55 + 5n)$$

$$2160 = n (55 + 5n)$$

$$= 5n^2 + 55n$$
Therefore:

\[5n^2 + 55n - 2160 = 0\]

\[n^2 + 11n - 432 = 0\] (divide by 5)

\[(n - 16)(n + 27) = 0\]

Therefore, the two possible solutions are \(n = 16\) and \(n = -27\). However, since \(n\) must be positive, \(n = 16\) is the only valid solution. Therefore, the runner must run for 16 weeks to complete 1 080 km.

6. The formula for the sum of the first 10 terms is:

\[S_{10} = \frac{n}{2} (2a + (n - 1)d)\]

\[145 = 5(2a + 9d)\]

\[29 = 2a + 9d\]

You also know the following:

\[t_3 = a + 2d\]

\[t_4 = a + 3d\]

\[t_9 = a + 8d\]

Therefore:

\[5t_3 = t_4 + t_9\]

\[5(a + 2d) = a + 3d + a + 8d\]

\[5a + 10d = 2a + 11d\]

\[3a = d\]

Therefore, you can substitute the value of \(d\) into the first equation, and calculate the value of \(a\):

\[29 = 2a + 9d\]

\[29 = 2a + 9(3a)\]

\[29 = 29a\]

\[a = 1\]

Now you can calculate \(d\):

\[d = 3a\]

\[= 3\]

Therefore, the first three terms of the sequence are 1, 4 and 7.
7. The first term of the sequence is $11p \, (a)$. The common difference for the sequence is:

\[ d = 13p - 11p \]

\[ = 2p \]

Therefore, you can use the general formula for the $n$th term, and solve for $n$ to find which term corresponds to $243p$.

\[ t_n = a + (n - 1)d \]

\[ 243p = 11p + (n - 1)2p \]

\[ = 11p + 2np - 2p \]

\[ = 9p + 2np \]

\[ 234p = 2np \]

\[ 117 = n \]

Therefore, the 117th term of the series is $243p$.

8. The value of the tenth term is equal to:

\[ S_{10} - S_9 = 10(10 + 2) - 9(9 + 2) \]

\[ = 21 \]

9. (a) The general formula for the sum of the first eleven terms is:

\[ S_{11} = \frac{11}{2} (2a + 10d) \]

You also know that the first term, $a$, is $\log_{\frac{1}{2}} x$. The common difference ($d$) is:

\[ \log_{\frac{1}{2}} x^3 - \log_{\frac{1}{2}} x = 2 \log_{\frac{1}{2}} x \]

Therefore:

\[ 242 = \frac{11}{2} (2 \log_{\frac{1}{2}} x + 10 \times 2 \log_{\frac{1}{2}} x) \]

\[ = 11 \log_{\frac{1}{2}} x + 110 \log_{\frac{1}{2}} x \]

\[ = 121 \log_{\frac{1}{2}} x \]

\[ 2 = \log_{\frac{1}{2}} x \]

\[ x = \left(\frac{1}{2}\right)^2 \]

\[ = \frac{1}{4} \]
(b) The tenth term of the sequence is:

\[ t_{10} = a + (n - 1)d \]

\[ = \log_{\frac{1}{2}} \frac{1}{4} + 9 \times 2 \log_{\frac{1}{2}} \frac{1}{4} \]

\[ = 19 \log_{\frac{1}{2}} \frac{1}{4} \]

\[ = 19 \times 2 \]

\[ = 38 \]

The tenth term of the sequence is 38.

10. There are two arithmetic series in this question. Therefore, you need to work out the sum of each sequence, and then calculate the final answer.

The arithmetic series in the numerator has an initial value of 199, and a common difference of –2. You know that the last value is 101, therefore:

\[ t_n = a + (n - 1)d \]

\[ 101 = 199 - 2(n - 1) \]

\[ = 201 - 2n \]

\[ 2n = 100 \]

\[ n = 50 \]

Therefore:

\[ S_n = \frac{n}{2} (2a + (n - 1)d) \]

\[ = 25(398 + 49(-2)) \]

\[ = 7500 \]

The arithmetic series in the denominator has an initial value of 99, and a common difference of –2. You know that the last value is 1, therefore:

\[ t_n = a + (n - 1)d \]

\[ 1 = 99 - 2(n - 1) \]

\[ = 101 - 2n \]

\[ 2n = 100 \]

\[ n = 50 \]

Therefore:

\[ S_n = \frac{n}{2} (2a + (n - 1)d) \]

\[ = 25(198 + 49(-2)) \]

\[ = 2500 \]
Therefore:

\[
\frac{199 + 197 + 195 + \ldots + 101}{99 + 97 + 95 + \ldots + 1} = \frac{7500}{2500} = 3
\]

**Check your competence**

Now that you have worked through this lesson, please check that you can perform the tasks below:

- I can simplify expressions using the laws of exponents for rational exponents.
- I can add, subtract, multiply and divide simple surds.
- I can demonstrate an understanding of error margins.
- I can investigate number patterns, and hence:
  - make conjectures and generalisations; and
  - provide explanations and justifications, and attempt to prove conjectures.
- I can identify and solve problems involving number patterns, including, but not limited to arithmetic and geometric sequences and series.
- I can correctly interpret recursive formulae (for example, \( T_{n+1} = T_n + T_{n-1} \)).
- I can solve non-routine, unseen problems.

**The next lesson**

If you are sure that you understand the contents covered in this lesson, and have achieved your Assessment Standards, start Lesson 6.
Learning Outcomes for Lesson 6

After you have worked through Lesson 1, you should be able to do the following:

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<td>AS 11.1.2 (a) Simplify expressions using the laws of exponents for rational exponents.</td>
</tr>
<tr>
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<td>(b) Add, subtract, multiply and divide simple surds.</td>
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<td>(c) Demonstrate an understanding of error margins.</td>
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<tr>
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<td>AS 11.1.3 Investigate number patterns (including, but not limited to those where there is a constant second difference between consecutive terms in a number pattern, and the general term is therefore quadratic) and hence:</td>
</tr>
<tr>
<td></td>
<td>(a) make conjectures and generalisations</td>
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<tr>
<td></td>
<td>provide explanations and justifications, and attempt to prove conjectures.</td>
</tr>
<tr>
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<td>AS 11.1.6 Solve non-routine, unseen problems.</td>
</tr>
<tr>
<td></td>
<td>AS 12.1.3 (a) Identify and solve problems involving number patterns, including, but not limited to arithmetic and geometric sequences and series. Correctly interpret recursive formulae (for example, $T_{n+1} = T_n + T_{n-1}$).</td>
</tr>
<tr>
<td></td>
<td>AS 12.1.6 Solve non-routine, unseen problems.</td>
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We would like to help you to remain in complete control of your studies. So we give you an opportunity to check your competence at the end of this lesson.

Introduction

In Lesson 5, we introduced you to sequences and series. In particular, you learnt how to recognise:

- arithmetic sequences; and
- geometric sequences.
Lesson 5 concentrated on arithmetic sequences. In this lesson, we’ll investigate geometric sequences in more detail. The topics of this lesson are similar to those that you encountered in the previous lesson.

In Lesson 6, you will learn how to:

• determine the common ratio of a geometric sequence;
• determine the general term of a geometric sequence;
• determine the sum of \( n \) terms of a geometric sequence; and
• solve problems that include geometric sequences.

Finally, you’ll learn about convergent and divergent geometric series, including how to solve problems that include a convergent geometric series.

**The common ratio of a geometric sequence**

We begin by formally defining a geometric sequence.

If a sequence \( t_1; t_2; t_3; \ldots \) has the property that:

\[
\begin{align*}
t_2 &= t_1 \times r \\
t_3 &= t_2 \times r \\
t_4 &= t_3 \times r
\end{align*}
\]

And:

\[
t_n = t_{n-1} + r
\]

for a number \( r \) (which can be positive or negative), then such a sequence is a **geometric sequence** (GS).

The number \( r \) (which we multiply each term by to produce the next term) is known as the **common ratio**.

According to this definition:

\[
\frac{t_2}{t_1} = \frac{t_3}{t_2} = \frac{t_4}{t_3} \ldots = r
\]

The number \( r \) must remain constant over the entire length of the sequence.

Make sure that you clearly understand the difference between an AS and a GS. In an AS, we **add** the same number to each successive term. However, in a GS, we **multiply** each term by the same number.
**The general term of a geometric sequence**

Suppose that we have a GS with a first term \( t_1 \) equal to \( a \), and a common ratio of \( r \). We can then express the terms in the sequence as follows:

\[
\begin{align*}
    t_2 &= t_1 \times r \\
    &= ar \\
    t_3 &= t_2 \times r \\
    &= ar^2 \\
    t_4 &= t_3 \times r \\
    &= ar^3
\end{align*}
\]

Therefore, we can deduce the following formula for the general term of a geometric sequence:

\[
\sum_{n=1}^{\infty} t_n = ar^{n-1}
\]

where \( t_n \) is the \( n \)th term, or *general term*.

**The sum of the terms in a geometric series**

The next step is to deduce a formula for the sum \( S_n \) of the first \( n \) terms of a GS. Once again, suppose that we have a GS with a first term \( t_1 \) equal to \( a \), and a common ratio of \( r \).

We express the sum of the GS as follows:

\[
\sum_{n=1}^{n} S_n = t_1 + t_2 + \ldots + t_n
= a + ar + ar^2 \ldots + ar^{n-1}
\]

Now, if we multiply \( S_n \) by \( r \), we get:

\[
r \times S_n = ar + ar^2 + \ldots + ar^{n-1} + ar^n
\]

Finally, if we subtract \( r \times S_n \) from \( S_n \), then we have:

\[
a + ar + ar^2 + \ldots + ar^{n-1} - ar + ar^2 + \ldots + ar^{n-1} + ar^n
= a + 0 + 0 + \ldots + 0 - ar^n
\]
Therefore:

\[ S_n - r \times S_n = a - ar^n \]

\[ S_n(1 - r) = a - ar^n \]

\[ S_n = \frac{a(1 - r^n)}{1 - r} \]

Therefore, the general formula for the sum of \( n \) terms in a GS is:

\[ \sum S_n = \frac{a(1 - r^n)}{1 - r} \quad \text{or} \quad S_n = \frac{a(r^n - 1)}{r - 1} \]

Let's see how to use these formulas to solve problems that include a geometric sequence or series.

**EXAMPLE**

The third term of a GS is 3, and the seventh term is 48. Find the possible values of the sum of the first seven terms.

**Answer**

The general term of a GS is \( t_n = ar^{n-1} \).

Therefore, we know the following information:

\[ ar^2 = 3 \quad \text{(t}_3\text{)} \]

\[ ar^6 = 48 \quad \text{(t}_7\text{)} \]

From the first equation, we have:

\[ a = \frac{3}{r^2} \]

If we substitute this value for \( a \) into the second equation, we have:

\[ \frac{3}{r^2} \times r^6 = 48 \]

\[ 3r^4 = 48 \]

\[ r^4 = 16 \]

\[ r = \pm 2 \]

Therefore, the two values 2 and \(-2\) are both possible values for the common ratio.
We can calculate the value of the first term by substituting one of these values back into the first equation. Note that either choice of \( r \) gives the same answer for \( a \):

\[
a = \frac{3}{r^3}
\]

\[
= \frac{3}{4}
\]

Therefore, there are two possible geometric sequences that have a third term of 3, and a seventh term of 48. The sequence for a value of \( r \) equal to 2 is:

\[
\frac{3}{4}; \frac{3}{2}; 3; 6; \ldots
\]

The sum of the first seven terms for this sequence is:

\[
S_7 = \frac{a(1-r^n)}{1-r}
\]

\[
= \frac{3}{4} \times \frac{(1-2^7)}{1-2}
\]

\[
= \frac{3}{4} \times 127
\]

\[
= \frac{381}{4}
\]

The sequence corresponding to a value of \( r \) equal to \(-2\) is:

\[
\frac{3}{4}; -\frac{3}{2}; 3; -6; \ldots
\]

The sum of the first seven terms for this sequence is:

\[
S_7 = \frac{a(1-r^n)}{1-r}
\]

\[
= \frac{3}{4} \times \frac{(1-(-2)^7)}{1+2}
\]

\[
= \frac{3}{4} \times 43
\]

\[
= \frac{129}{4}
\]

In conclusion, there are two possible geometric sequences that have a third value of 3, and seventh value of 48.

The sum of the first seven terms of the first sequence is \( \frac{381}{4} \), and the sum of the first seven terms of the second sequence is \( \frac{129}{4} \).
Now let’s look at an example in which the terms contain variables.

**EXAMPLE**

Find the possible value(s) of $x$ if the terms $(x - 2)$, $(1 - 2x)$ and $(5x + 2)$ are consecutive terms in a GS.

**Answer**

Since the sequence is a GS, there must be a common ratio between each successive term.

Therefore:

$$\frac{1 - 2x}{x - 2} = \frac{5x + 2}{1 - 2x}$$

$$(1 - 2x)(1 - 2x) = (5x + 2)(x - 2)$$

$$1 - 4x + 4x^2 = 5x^2 - 10x + 2x - 4$$

$$x^2 - 4x - 5 = 0$$

$$(x - 5)(x + 1) = 0$$

$\therefore$ $x = 5$ or $x = -1.$

Now work through the following activity to practise what you have learnt about geometric sequences and series.

**Activity 9**

1. Identify which of the following sequences are geometric sequences (GS). For each GS, give the value of $r$, the common ratio.
   
   (a) 1; 5; 25; 125; . . .
   
   (b) $\frac{1}{2}$; $\frac{1}{4}$; $\frac{1}{8}$; $\frac{1}{16}$; . . .
   
   (c) $\sqrt{2}$; $\sqrt{8}$; $\sqrt{32}$; $\sqrt{50}$; . . .
   
   (d) $\frac{x}{y}$; $x$; $xy$; . . .
   
   (e) log 2; log 4; log 16; log 256; . . .

2. The $n$th term in the geometric sequence 3; 9; 27; . . . is 6 561. Solve for $n$, and so find the position of this term.

3. In a GS, the sixth term is 8, and the tenth term is 128. Find the sequence.
4. The third term of a GS is 4, and the sixth term is \( \frac{32}{27} \). Find an expression for the \( n \)th term of this GS.

5. Determine \( S_8 \) for the GS \( 4; -2; 1; -\frac{1}{2}; \ldots \).

6. How many terms of the series \( 3; 6; 12; \ldots \) must be added to give a sum of \( 3069 \)?

7. In a finite GS, the sum to \( n \) terms is 728. The first term is 2, and the last term is 486.
   (a) Find the common ratio (\( r \)).
   (b) Find the number of terms in the sequence.

Answers

1. In order for a sequence to be a GS, there must be a constant ratio between successive terms. This implies that, if a sequence is a GS, then:

\[
\frac{t_2}{t_1} = \frac{t_3}{t_2} = \frac{t_4}{t_3} = \ldots = r
\]

So you can check if a sequence is a GS by calculating these ratios. If the ratio is a constant value, the sequence is a GS.

(a) \[
\frac{t_2}{t_1} = \frac{5}{1} = 5
\]

(b) \[
\frac{t_2}{t_1} = \frac{-\frac{1}{2}}{1} = -\frac{1}{2}
\]

Therefore, this is a GS, with a common ratio equal to 5.
\[ t_4 = \frac{1}{16} \]

\[ t_3 = \frac{1}{8} \]

\[ t_2 = \frac{1}{4} \]

Therefore, this is a GS, with a common ratio equal to \(-\frac{1}{2}\).

(c) \[
\frac{t_2}{t_1} = \frac{2\sqrt{2}}{\sqrt{2}} = 2
\]

\[
\frac{t_3}{t_2} = \frac{4\sqrt{2}}{2\sqrt{2}} = 2
\]

\[
\frac{t_4}{t_3} = \frac{5\sqrt{2}}{4\sqrt{2}} = \frac{5}{4} \neq 2
\]

Therefore, this is not a GS.

(d) \[
\frac{t_2}{t_1} = \frac{x}{y} = \frac{xy}{x}
\]

\[
\frac{t_3}{t_2} = \frac{xy}{x}
\]

Therefore, this is a GS, with a common ratio equal to \(\frac{xy}{x}\).

(e) \[
\frac{t_2}{t_1} = \frac{2\log 2}{\log 2} = 2
\]

\[
\frac{t_3}{t_2} = \frac{4\log 2}{2\log 2} = 2
\]
\[
\frac{t_4}{t_3} = \frac{8 \log 2}{4 \log 2} = 2
\]

Therefore, this is a GS, with a common ratio equal to 2.

2. The formula for the \(n\)th term in a GS is:
\[
t_n = ar^{n-1}
\]

You need to solve for \(n\). Therefore, you first need to work out the values of \(a\) and \(r\) in this particular sequence. The first term of the sequence is 3, so this is the value of \(a\). You can calculate \(r\) as follows:
\[
r = \frac{t_2}{t_1} = \frac{9}{3} = 3
\]

Therefore, you need to solve for \(n\) in the following equation:

\[
6561 = 3 \times 3^{n-1}
\]
\[
2187 = 3^{n-1}
\]
\[
37 = 3^{n-1}
\]

Therefore:
\[
\begin{align*}
  n - 1 &= 7 \\
  n &= 8
\end{align*}
\]

Therefore, 6561 is the eighth term in the sequence.

3. The sixth term is 8. Therefore:
\[
ar^5 = 8
\]
\[
a = \frac{8}{r^5}
\]
The tenth term is 128. Therefore:
\[
ar^9 = 128
\]
Substitute the value of \(a\) from the first equation into the second equation:
\[
\frac{8}{r^5} \times r^9 = 128
\]
\[
8r^4 = 128
\]
\[
r^4 = 16
\]
\[
r = ±2
\]
Since there are two possible values of $r$, there are also two possible values for $a$. Therefore, if $r$ is equal to 2, then $a$ is:

\[
a = \frac{8}{r^5} = \frac{8}{32} = \frac{1}{4}
\]

If $r$ is equal to $-2$, then $a$ is:

\[
a = \frac{8}{r^5} = \frac{8}{-32} = -\frac{1}{4}
\]

So there are two possible sequences. If $r = 2$, then the sequence is:

\[
\frac{1}{4}; \frac{1}{2}; 1; 2; \ldots
\]

If $r = -2$, then the sequence is:

\[
-\frac{1}{4}; \frac{1}{2}; -1; 2; \ldots
\]

Note that, in both sequences, the sixth term is 8, and the tenth term is 128.

4. The general term of a GS is $t_n = ar^{n-1}$. Therefore, we know the following information:

\[
ar^2 = 4 \quad (t_3)
\]

\[
ar^5 = \frac{32}{27} \quad (t_6)
\]

From the first equation, we have:

\[
a = \frac{4}{r^2}
\]

If we substitute this value for $a$ into the second equation, we have:

\[
\frac{4}{r^2} \times r^5 = \frac{32}{27}
\]

\[
4r^3 = \frac{32}{27}
\]

\[
r^3 = \frac{8}{27} = \frac{2^3}{3^3}
\]
You can calculate the value of the first term by substituting $\frac{2}{3}$ back into the first equation:

\[
a = \frac{4}{r^2} = 4 \times \left(\frac{3}{2}\right)^2 = 4 \times \frac{9}{4} = 9
\]

Therefore, the expression for the $n$th term is:

\[
t_n = 9 \times \left(\frac{2}{3}\right)^{n-1}
\]

5. The formula for the sum of eight terms of a GS is:

\[
S_8 = \frac{a(1-r^8)}{1-r}
\]

Therefore, you first need to work out the values of $a$ and $r$ before you can calculate the sum of the terms. The first term is 4, so this is the value of $a$. You can calculate $r$ as follows:

\[
r = \frac{t_2}{t_1} = -\frac{2}{4} = -\frac{1}{2}
\]

Therefore:

\[
S_8 = \frac{a(1-r^8)}{1-r} = \frac{4(1-\left(-\frac{1}{2}\right)^8)}{1+\frac{1}{2}} = 2,656
\]
6. The general formula for the sum of the terms of a GS is:

\[ S_n = \frac{a(1 - r^n)}{1 - r} \]

Therefore, you first need to work out the values of \( a \) and \( r \), and then solve for \( n \). The first term is 3, so this is the value of \( a \). You can calculate \( r \) as follows:

\[ r = \frac{t_2}{t_1} = \frac{6}{3} = 2 \]

Therefore:

\[ 3069 = \frac{a(1 - r^n)}{1 - r} = \frac{3(1 - 2^n)}{1 - 2} = \frac{3 - 3 \times 2^n}{-1} = -3 + 3 \times 2^n \]

\[ 3072 = 3 \times 2^n \]

\[ 1024 = 2^n \]

\[ 2^{10} = 2^n \]

Therefore, \( n \) is equal to 10, and the sum of the first 10 terms is 3069.

7. (a) You know that \( S_n = 728 \), and that \( a \) is equal to 2. Therefore:

\[ \frac{2(1 - r^n)}{1 - r} = 728 \]

You also know that the last term is 486. Therefore:

\[ 486 = t_n = 2 \times r^{n-1} \]

\[ 243 = r^{n-1} \]

\[ \frac{r^n}{r} = 243r = r^n \]
You can now substitute the value for \( r^n \) into the first equation, which gives:

\[
\frac{2(1 - 243r)}{1 - r} = 728
\]

\[
2(1 - 243r) = 728(1 - r)
\]

\[
2 - 486r = 728 - 728r
\]

\[
242r = 726
\]

\[
r = 3
\]

(b) You can use the fact that you know that the last term is 486 to calculate the number of the terms in the sequence.

\[
486 = t_n
\]

\[
= 2 \times r^{n-1}
\]

You know that \( r \) is equal to 3. Therefore:

\[
486 = 2 \times 3^{n-1}
\]

\[
243 = 3^{n-1}
\]

\[
3^5 = 3^{n-1}
\]

Therefore:

\[
n - 1 = 5
\]

\[
n = 6
\]

Therefore, there are six terms in the sequence.

If you are asked to find the general term \((T_n)\) of a sequence, you must first determine whether it is:

- linear (arithmetic) – the first difference is a constant;
- quadratic – the second difference is a constant; or
- geometric – there is a common ratio.

**If the sequence is linear (arithmetic)**

If the sequence is linear (arithmetic) \( T_n = an + c \), then we use the following formula:

\[
T_n = a + (n - 1)d
\]
where:

- $a$ represents the first;
- $n$ represents the position of the term or the number of terms; and
- $d$ represents the first difference (it must be a constant).

Let’s work through an example of what to do if the sequence is linear (arithmetic).

**EXAMPLE**

Find the general term of the sequence 2 ; 5 ; 8 ; 11 ; 14 ;

**Answer**

The difference between any two consecutive terms is 3. That means the first difference is 3. Therefore the sequence is linear.

From the above $a = 2$ ; $n = ?$ ; $d = 3$

$\therefore T_n = a + (n - 1)d$

$\therefore T_n = 2 + (n - 1)3$

$\therefore T_n = 2 + 3n - 3$

$\therefore T_n = 3n - 1$

Therefore, the general term is:

$T_n = 3n - 1$

If you have the general term of a sequence, then you can find a term in any position. For example, if you want to find the tenth term, we find $T_{10}$:

$\therefore T_{10} = 3(10) - 1$

$\therefore T_{10} = 29$

Now let’s look at what to do if the sequence is quadratic.

**If the sequence is quadratic**

If the sequence is quadratic, $T_n = an^2 + bn + c$, we then use the formula:

\[
a + b + c = \text{the first term of the sequence}
\]

\[
3a + b = \text{the first value of the first difference}
\]

\[
2a = \text{the second difference}
\]

Let’s work through an example of how to deal with a quadratic sequence.
EXAMPLE

Find the general term of the sequence 0; 9; 22; 39 ; 60;

Answer

The first difference between two consecutive terms are:

9 13 17 21, which is not constant.

So, we proceed to find the second difference:

4 4 4, which is constant.

Therefore, the sequence is quadratic (because the second difference is constant).

Therefore:

\[ a + b + c = 0 \]
\[ 3a + b = 9 \]
\[ 2a = 4 \]

So:

\[ 2a = 4 \] (divide by 2)
\[ \therefore a = 2 \]

To find \( b \), we substitute \( a = 2 \) into \( 3a + b = 9 \).

\[ \therefore 3(2) + b = 9 \]
\[ \therefore 6 + b = 9 \]
\[ \therefore b = 9 - 6 \]
\[ \therefore b = 3 \]

To find \( c \), we substitute \( a = 2 \) and \( b = 3 \) into \( a + b + c = 0 \)

\[ \therefore 2 + 3 + c = 0 \]
\[ \therefore c = 0 - 2 - 3 \]
\[ \therefore c = - 5 \]

Therefore the general term is \( T_n = 2n^2 + 3n - 5 \)

Now you can find a term in any position. For example, if we determine the tenth term, we have:

\[ T_n = 2n^2 + 3n - 5 \]
\[ T_{10} = 2(10)^2 + 3(10) - 5 \]
\[ T_{10} = 200 + 30 - 5 \]
\[ T_{10} = 225 \]

Now let’s look at what to do if the sequence is geometric.
**If the sequence is geometric**

If the sequence is geometric, we use the following formula:

\[ T_n = ar^{n-1} \]

Where:

- \( a \) represents the first;
- \( n \) represents the position of the term or the number of terms; and
- \( r \) represents the common ratio (a constant that each term is multiplied by to get the next term).

To find \( r \), we can use the following formula:

\[ r = \frac{T_n}{T_{n-1}} = \frac{T_{n+1}}{T_n} \]

For example:

\[ r = \frac{T_2}{T_1} = \frac{T_3}{T_2} \]

Now let’s work through an example of what to do if the sequence is geometric.

**EXAMPLE**

Find the general term of the sequence 3 ; 6 ; 12 ; 24 ; 48 ;........

**Answer**

To find the common ratio we use the following formula:

\[ r = \frac{T_2}{T_1} = \frac{T_3}{T_2} \]

\[ r = \frac{6}{3} = \frac{12}{6} = 2 \]

Therefore:

\[ a = 3, r = 2, n = ? \]

\[ :: T_n = ar^{n-1} \]

\[ :: T_n = (3)(2)^{n-1} \]
Now you can find a term in any position. For example, if we determine the fifth term, we have:

\[
T_n = (3)(2)^{n-1}
\]

\[
T_5 = (3)(2)^4
\]

\[
T_5 = (3)(16)
\]

\[
T_5 = 48
\]

Now work carefully through Activity 9 to practise what you have learnt so far.

### Activity 10

Find the \( n \)th term (general term) for each of the following sequences:

1. 4; 8; 12; 16; 20; . . .
2. 3; 1; -1; -3; -5; . . .
3. 2; -4; 8; -16; 32; . . .
4. \( \frac{1}{4} \); 1; 4; 16; 64; . . .
5. 2; 7; 14; 23; 34; . . .
6. 5; 12; 25; 44; 69; . . .

**Answer**

1. The first difference between any two consecutive terms is 4.

Therefore, the sequence is linear and we use the formula \( T_n = a + (n-1)d \).

So:

\[
a = 4, \; n = ?, \; d = 4,
\]

\[
\therefore T_n = a + (n-1)d
\]

\[
\therefore T_n = 4 + (n-1)4
\]

\[
\therefore T_n = 4n - 4
\]

\[
\therefore T_n = 4n
\]

2. The first difference between any two consecutive terms is -2.

Therefore, the sequence is linear and we use the formula \( T_n = a + (n-1)d \).

\[
a = 3, \; n = ?, \; d = -2
\]

\[
\therefore T_n = a + (n-1)d
\]

\[
\therefore T_n = 3 + (n-1)(-2)
\]

\[
\therefore T_n = 3 - 2n + 2
\]

\[
\therefore T_n = -2n + 5
\]
3. If we use the formula \( r = \frac{T_2}{T_1} = \frac{T_3}{T_2} \), we note that:

\[ r = \frac{-4}{2} = \frac{8}{-4} = -2 \]

Therefore, the sequence is geometric and we use the following formula:

\[ \therefore T_n = ar^{n-1} \]

\[ a = 2, \ n = ?, \ r = -2 \]

\[ \therefore T_n = ar^{n-1} \]

\[ \therefore T_n = (2)(-2)^{n-1} \]

4. \( r = \frac{4}{1} = 4 \)

Therefore, the sequence is geometric and we use the following formula:

\[ \therefore T_n = ar^{n-1} \]

So:

\[ a = \frac{1}{4}, \ n = ?, \ r = 4 \]

\[ \therefore T_n = ar^{n-1} \]

\[ \therefore T_n = \left(\frac{1}{4}\right)(4)^{n-1} \]

\[ \therefore T_n = (2^{-2})(4^{n-1}) \]

\[ \therefore T_n = (2^{-2})(2^{2n-2}) \]

\[ \therefore T_n = 2^{2n-4} \]

5. The first difference is: 5 7 9 11.

Therefore the first difference is not constant. If we determine the second difference, we have:

2 2 2

Therefore the second difference is 2, which means the sequence is quadratic.

So:

\[ \therefore a + b + c = 2 \]

\[ 3a + b = 5 \]

\[ 2a = 2 \]
So:
\[ 2a = 2 \]  
\[ \therefore a = 1 \]

To find \( b \), we substitute \( a = 1 \) into \( 3a + b = 5 \).
\[ \therefore 3(1) + b = 5 \]
\[ \therefore 3 + b = 5 \]
\[ \therefore b = 5 - 3 \]
\[ \therefore b = 2 \]

To find \( c \), we substitute \( a = 1 \) and \( b = 2 \) into \( a + b + c = 2 \)
\[ \therefore 1 + 2 + c = 2 \]
\[ \therefore c = 2 - 1 - 2 \]
\[ \therefore c = -1 \]

Therefore, the general term is \( T_n = 1n^2 + 2n - 1 \) or \( T_n = n^2 + 2n - 1 \)

6. The first difference is:

\[ 7 \quad 13 \quad 19 \quad 25 \]
Therefore, the first difference is not constant. If we determine the second difference, we have:

\[ 6 \quad 6 \quad 6 \]

Therefore, the second difference is 6, which means the sequence is quadratic.
\[ \therefore a + b + c = 5 \]
\[ 3a + b = 7 \]
\[ 2a = 6 \]

So:
\[ 2a = 6 \]  
\[ \therefore a = 3 \]

To find \( b \), we substitute \( a = 3 \) into \( 3a + b = 7 \).
\[ \therefore 3(3) + b = 7 \]
\[ \therefore 9 + b = 7 \]
\[ \therefore b = 7 - 9 \]
\[ \therefore b = -2 \]

To find \( c \), we substitute \( a = 3 \) and \( b = -2 \) into \( a + b + c = 5 \)
\[ \therefore 3 + (-2) + c = 5 \]
\[ \therefore c = 5 - 3 + 2 \]
\[ \therefore c = 4 \]

Therefore, the general term is \( T_n = 3n^2 - 2n + 4 \)
In the final section of this lesson, we’ll consider a special kind of geometric series, called a *convergent geometric series*.

**Convergent geometric series**

The word *convergent* comes from the word *converge*.

The Oxford English Dictionary defines *converge* as a verb that means ‘to come together from different directions so as eventually to meet’.

So, convergent describes something that converges, or comes together to meet. The sum of a geometric series is the value that you get after adding together the terms in the series. A convergent geometric series is special type of GS. When we say that the series converges, we mean that the sum of the series becomes closer and closer to a certain value, as the number of terms in the sum increases.

Let’s play a mind game. Imagine that you are standing in front of a wall, and that you are exactly 2 m away from the wall. Now imagine that you are going to walk towards the wall, but that each step will be exactly half the remaining distance between you and the wall. So, your first step towards the wall is exactly 1 m in length. Your next step is half a metre, your third step is a quarter or a metre, and so on.

It probably not take you very long to realise that, no matter how many steps you take towards the wall, you will never actually reach the wall! We can represent the size of the steps you are taking in this mind game by the following geometric series:

\[ 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots \]

*Table 7* shows the data for the first five terms in the series.

<table>
<thead>
<tr>
<th>n</th>
<th>( T_n )</th>
<th>( S_n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>( \frac{1}{2} ) = 0,5</td>
<td>1,5</td>
</tr>
<tr>
<td>3</td>
<td>( \frac{1}{4} ) = 0,25</td>
<td>1,75</td>
</tr>
<tr>
<td>4</td>
<td>( \frac{1}{8} ) = 0,125</td>
<td>1,875</td>
</tr>
<tr>
<td>5</td>
<td>( \frac{1}{16} ) = 0,0625</td>
<td>1,937 5</td>
</tr>
</tbody>
</table>

As you can see, as \( n \) gets larger, the value of \( T_n \) gets smaller, and \( S_n \) gets closer and closer to 2.

*We say that* \( S_n \) *tends towards 2, but it will never actually reach 2.*

Therefore, we say the series converges to 2, and that the sum of an infinite number of terms is 2.
Let’s return to the formula for the sum of the first \( n \) terms of a geometric sequence \( a + ar + ar^2 + \ldots \). The formula is:

\[
S_n = \frac{a(1 - r^n)}{1 - r}
\]

Suppose that in an infinite geometric sequence, \(-1 < r < 1\). In such a case, as the value of \( n \) increases, the number \( r^n \), which appears in the numerator, will approach 0. We can get \( r^n \) as close to 0 as we wish by simply taking a large enough value of \( n \).

For example, assume that \( r = \frac{1}{2} \), as was the case in the example in Table 7. Therefore, in the formula, \( r^n = \left(\frac{1}{2}\right)^n \).

Table 8 shows how the value of \( r^n \) becomes smaller and smaller in value as \( n \) increases.

<table>
<thead>
<tr>
<th>( N )</th>
<th>( r^n )</th>
<th>Decimal value of ( r^n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( \left(\frac{1}{2}\right)^1 )</td>
<td>0,5</td>
</tr>
<tr>
<td>2</td>
<td>( \left(\frac{1}{2}\right)^2 )</td>
<td>0,25</td>
</tr>
<tr>
<td>3</td>
<td>( \left(\frac{1}{2}\right)^3 )</td>
<td>0,125</td>
</tr>
<tr>
<td>4</td>
<td>( \left(\frac{1}{2}\right)^4 )</td>
<td>0,0625</td>
</tr>
<tr>
<td>5</td>
<td>( \left(\frac{1}{2}\right)^5 )</td>
<td>0,03125</td>
</tr>
</tbody>
</table>

The tendency for \( r^n \) to move closer and closer to 0 as \( n \) increases is expressed mathematically as follows:

\[
\lim_{n \to \infty} r^n = 0
\]

The way to read this notation is to say that ‘the limit of \( r^n \) as \( n \) approaches infinity is 0.’ Keeping this in mind, let’s see what happens to \( S_n \) as \( n \) increases. As \( n \) increases, the value of \( r^n \) becomes closer and closer to 0.
Therefore, we can say that, as \( n \) increases, the value of \( S_n \) will approach \( \frac{a(1-0)}{1-r} \),
which can be simplified to \( \frac{a}{1-r} \).

Mathematically, we can express this as follows:

\[
\lim_{n \to \infty} S_n = \frac{a}{1-r}
\]

Or:

\[
S_n = \frac{a}{1-r}
\]

In the second form of the statement, we read \( S_n \) as ‘the sum to infinity’. The tendency for \( r^n \) to approach 0 is true for any value of \( r \) between \(-1\) and \( 1 \).

A geometric sequence with \(-1 < r < 1\), \( r \neq 0 \) is a convergent sequence. In other words, the sequence converges on a particular value.

What about geometric sequences with \( r < -1 \) or \( r > 1 \)?

In sequences in which \( r < -1 \) or \( r > 1 \), we find that \( S_n \) does not approach a definite limit as \( n \) increases. In such a case, the sequence diverges (or the sequence is divergent).

For example, the following series is a divergent geometric series:

\[
2 + 5 + 8 + 11 + \ldots
\]

Table 9 shows the data for the first five terms in the series.

<table>
<thead>
<tr>
<th>( n )</th>
<th>( T_n )</th>
<th>( S_n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>5</td>
<td>7</td>
</tr>
<tr>
<td>3</td>
<td>8</td>
<td>15</td>
</tr>
<tr>
<td>4</td>
<td>11</td>
<td>26</td>
</tr>
<tr>
<td>5</td>
<td>14</td>
<td>40</td>
</tr>
</tbody>
</table>

As you can see from Table 9, as \( n \) gets larger, \( T_n \) gets larger, and \( S_n \) grows very rapidly. The sum of an infinite number of terms would be \( \infty \). We say that the series diverges. Here the sum to infinity cannot be calculated because as we add values we will either get a very large value (tends to infinity) or a very small value (tends to negative infinity). This is an example where the value cannot be determined.

Let’s work through an example to see how we can make use of what we know about a convergent series.
EXAMPLE

1. Find the sum to infinity of the sequence $2 + 0,4 + 0,08 + \ldots$

2. Determine the number of terms required for the difference between the sum and the sum to infinity to be less than 0,001.

Answer

1. The value of $a$ in the GS is 2. We can calculate $r$ as follows:

$$r = \frac{t_2}{t_1}$$

$$= \frac{0,4}{2}$$

$$= 0,2$$

Therefore:

$$S_\infty = a \frac{1}{1 - r}$$

$$= 2 \frac{1}{1 - 0,2}$$

$$= 2 \frac{1}{0,8}$$

$$= 2,5$$

2. We need the following inequality to be true:

$$S_\infty - S_n < 0,001$$

The formula for $S_n$ is:

$$S_n = \frac{2(1 - 0,2^n)}{0,8}$$

Therefore, the original inequality becomes:

$$2,5 - \frac{2(1 - 0,2^n)}{0,8} < 0,001$$

$$2 - 2 + 2(0,2)^n < 0,000\,8$$  \hspace{1cm} (multiply both sides by 0,8)

$$(0,2)^n < 0,000\,4$$

$$n \log 0,2 < \log 0,000\,4$$

$$n > \frac{\log 0,000\,4}{\log 0,2}$$

$$> 4,86$$

Therefore, since $n$ must be a whole number, $n$ must be greater than or equal to 5 for the difference between the sum and the sum to infinity to be less than 0,001.
Now let’s look at another example of a convergent geometric series, but this time including variables.

**EXAMPLE**

For which values of $x$ will the following sequence converge?

$$x + \frac{x^2}{1-x} + \frac{x^3}{(1-x)^2} + \ldots$$

**Answer**

The common ratio for the series is $\frac{x}{1-x}$.

We know that for a series to converge, we must have $-1 < \frac{x}{1-x} < 1$.

\[
\begin{align*}
\frac{x}{1-x} &< 1 \quad \text{and} \quad \frac{x}{1-x} > -1 \\
\frac{x}{1-x} &< 1 \quad \text{and} \quad \frac{x}{1-x} > -1 \\
\frac{x-(1-x)}{1-x} &< 0 \quad \text{and} \quad \frac{x+1-x}{1-x} > 0 \\
\frac{2x-1}{1-x} &< 0 \quad \text{and} \quad \frac{1}{1-x} > 0 \\
\frac{2x-1}{x-1} &< 0 \quad \text{and} \quad 1-x > 0 \\
\frac{2x-1}{x-1} &> 0 \quad \text{and} \quad x-1 < 0 \quad \text{(Sign changes when we multiply by a negative value)}
\end{align*}
\]

\[\therefore x < \frac{1}{2} \quad \text{or} \quad x > 1 \quad \text{and} \quad x < 1.\]

The combined solution: \[\therefore x < \frac{1}{2} \quad \text{(must satisfy both)}\]

Remember, to solve the inequality $\frac{2x-1}{x-1} > 0$, we make use of Lesson 2 of the Grade 11 Study Unit Functions and Algebra: Simultaneous Equations, Inequalities and Linear Programming.

Therefore, the sequence will converge for any value of $x < \frac{1}{2}$.

Now work carefully through the following activity to practise what you have learnt about convergent geometric series.
Activity 11

1. Determine which of the following sequences are convergent geometric sequences:
   
   (a) 4; 2; 1; \( \frac{1}{2} \) . . .
   
   (b) 4; 2; 0; -2; . . .
   
   (c) 4; -2; 1; -\( \frac{1}{2} \) . . .
   
   (d) \( t_n = 3 \times \left( \frac{1}{3} \right)^n \)
   
   (e) \( \log 2; \log 4; \log 16; . . . \)

2. Determine \( S_\infty \) for each convergent GS in Question 1 above.

3. In a GS, \( r = \frac{3}{5} \) and \( S_\infty = \frac{4}{6} \). Find the sequence.

4. In a convergent GS, \( S_5 = 31 \) and \( S_\infty = 32 \). Find the sequence.

5. Prove that the series \( 1 + \frac{2x}{3 + x^2} + \left( \frac{2x}{3 + x^2} \right)^2 + . . . \) converges for all real values of \( x \).

6. A ball falls from a height of 10 m. Each time it bounces back up, it reaches 80% of the previous height.
   
   (a) Calculate the height reached by the ball on the first and second bounces.
   
   (b) Calculate the total vertical distance travelled by the ball when it finally comes to rest.

7. Express 0.7 as an ordinary fraction by using an infinite GS.

Answers

1. The value of \( r \) must be between -1 and 1 in order for a geometric series to be convergent. Therefore, there are two steps required for each of these questions. The first is to establish if the series is a geometric series at all. Then, work out the value of \( r \), and determine if it falls between -1 and 1. If both of these criteria are true, then the series is a convergent geometric series.

   (a) \( \frac{t_2}{t_1} = \frac{2}{4} \)

      \[ = \frac{1}{2} \]

      \[ \frac{t_3}{t_2} = \frac{1}{2} \]
\[
\frac{t_4}{t_3} = \frac{\frac{1}{2}}{1} = \frac{1}{2}
\]

Therefore, this is a GS, with a common ratio equal to \(\frac{1}{2}\). The series also converges, since \(-1 < r < 1\).

(b) \[
\frac{t_2}{t_1} = \frac{\frac{2}{4}}{1} = \frac{1}{2}
\]

\[
\frac{t_3}{t_2} = \frac{0}{2} = 0
\]

\[
\frac{t_4}{t_3} = \frac{-2}{0} = \text{undefined}
\]

Therefore, this is a not GS.

(c) \[
\frac{t_2}{t_1} = \frac{-2}{4} = -\frac{1}{2}
\]

\[
\frac{t_3}{t_2} = \frac{-\frac{1}{2}}{2} = -\frac{1}{2}
\]

\[
\frac{t_4}{t_3} = \frac{-\frac{1}{2}}{1} = -\frac{1}{2}
\]

Therefore, this is a GS, with a common ratio equal to \(-\frac{1}{2}\). The series also converges, since \(-1 < r < 1\).
(d) The first series is:

\[ 3 \times \left( \frac{1}{3} \right)^1 + 3 \times \left( \frac{1}{3} \right)^2 + 3 \times \left( \frac{1}{3} \right)^3 + 3 \times \left( \frac{1}{3} \right)^4 + \ldots \]

Or:

\[ 1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} \ldots \]

Therefore:

\[ \frac{t_2}{t_1} = \frac{\frac{1}{3}}{1} \]

\[ = \frac{1}{3} \]

\[ \frac{t_3}{t_2} = \frac{\frac{1}{9}}{\frac{1}{3}} \]

\[ = \frac{1}{3} \]

\[ \frac{t_4}{t_3} = \frac{\frac{1}{27}}{\frac{1}{9}} \]

\[ = \frac{1}{3} \]

Therefore, this is a GS, with a common ratio equal to \( \frac{1}{3} \). The series also converges, since \(-1 < r < 1\).

(e) You can rewrite the series as:

\[ \log 2 + 2 \log 2 + 4 \log 2 + \ldots \]

Therefore:

\[ \frac{t_2}{t_1} = \frac{2 \log 2}{\log 2} \]

\[ = 2 \]

\[ \frac{t_3}{t_2} = \frac{4 \log 2}{2 \log 2} \]

\[ = 2 \]

Therefore, this is a GS, with a common ratio equal to 2. The series does not converge, since \( r > 1 \).
2. The sum to infinity for Question 1(a) is:

\[
S_\infty = \frac{a}{1-r} = \frac{4}{1-\frac{1}{2}} = 8
\]

The sum to infinity for Question 1(c) is:

\[
S_\infty = \frac{a}{1-r} = \frac{1}{1-\left(-\frac{1}{2}\right)} = 2\frac{2}{3}
\]

The sum to infinity for Question 1(d) is:

\[
S_\infty = \frac{a}{1-r} = \frac{4}{1-\frac{1}{3}} = \frac{3}{2}
\]

3. You know that \(r\) is equal to \(\frac{3}{5}\), and that \(S_\infty\) is \(4\frac{1}{6}\). You need to calculate the value of \(a\) in order to work out the sequence. You can do so by substituting the value of \(r\) into the formula for \(S_\infty\), and then solving for \(a\).

The calculation is as follows:

\[
S_\infty = \frac{a}{1-r} = \frac{4\frac{1}{6}}{1-\frac{3}{5}} = \frac{25}{6} = \frac{a}{\frac{2}{5}} \Rightarrow \frac{25}{6} \times \frac{2}{5} = a \quad \text{(multiply both sides by \(\frac{2}{5}\))}
\]

\[
a = \frac{5}{3}
\]

Therefore, the sequence is \(\frac{5}{3}; 1; \frac{3}{5}; \frac{9}{25}; \ldots\)
4. You know that $S_5 = 31$. Therefore:

\[
\frac{a(1-r^5)}{1-r} = 31
\]

You also know that $S_\infty = 32$. Therefore:

\[
\frac{a}{1-r} = 32
\]

\[a = 32(1 - r)\]

Substitute the value of $a$ from the second equation into the first equation:

\[
\frac{32(1-r)(1-r^5)}{1-r} = 31
\]

\[32(1 - r^5) = 31\]

\[1 - r^5 = \frac{31}{32}\]

\[r^5 = 1 - \frac{31}{32}\]

\[= \frac{1}{32}\]

\[= \frac{1}{2^5}\]

\[= \left(\frac{1}{2}\right)^5\]

Therefore, $r$ is equal to $\frac{1}{2}$. Now you can calculate $a$:

\[a = 32(1 - r)\]

\[= 32 \times \frac{1}{2}\]

\[= 16\]

Therefore, the sequence is:

16; 8; 4; . . .
5. In order to prove that the series converges, you need to prove that the value of $r$ lies between $-1$ and $1$, for all real values of $x$. In other words, you need to prove that $-1 < r < 1$.

So, you first need to calculate the value of $r$:

$$
\frac{t_2}{t_1} = \frac{2x}{3 + x^2}
$$

The method that you can use to prove that $\frac{2x}{3 + x^2} < 1$ is to assume that it is true, and then rewrite the inequality in a form that either confirms or invalidates the initial assumption. In its current form, it’s not possible to say for sure if the statement is true or false. Therefore, assume that the inequality is true:

$$
-1 < \frac{2x}{3 + x^2} < 1
$$

$$
\frac{2x}{3 + x^2} < 1 \quad \text{and} \quad \frac{2x}{3 + x^2} > -1
$$

$2x < 3 + x^2$ and $2x > -(3 + x^2)$ (you can multiply both sides by $(3 + x^2)$ with changing the inequality sign since $(3 + x^2) > 0$

$$
\therefore -x^2 + 2x - 3 < 0 \quad \text{and} \quad 2x > -3 - x^2
$$

$$
\therefore x^2 - 2x + 3 > 0 \quad \text{and} \quad x^2 + 2x + 3 > 0
$$

Both inequalities are true for all values of $x$. (you can check both inequalities on your make completing the square.) Therefore, the series converges.

6. (a) Note that 80% is equal to 0.8. The height that the ball bounces back the first time is:

$$
10 \times 0.8 = 8
$$

Therefore, the ball bounces back up to a height of 8 m the first time. The height that the ball bounces back the second time is:

$$
8 \times 0.8 = 6.4
$$

Therefore, the ball bounces back up to a height of 6.4 m after the second bounce.

(b) In order to answer this question, you need to work out the geometric series that the distance travelled by the bouncing ball represents. The series is:

$$
10 + 0.8 + 0.8 + 6.4 + 6.4 + 5.12 + 5.12 + \ldots
$$

The reason that some of the values repeat in the series is that the first value shows how far up the ball bounces up, and the second value represents the ball falling that same distance back to the ground, before bouncing again.
Therefore, you can rewrite the series as:

\[ 10 + (16 + 12.8 + 10.24 + \ldots) \]

The values in the brackets represent a geometric series with \( a = 16 \), and \( r = 0.8 \). Since you don't know how many times the ball bounces, you can use the formula for \( S_\infty \) to calculate the total distance travelled by the ball.

\[
S_\infty = 10 + \frac{16}{1-0.8} \\
= 10 + 80 \\
= 90
\]

Therefore, the ball travels a total of 90 m before it finally comes to rest.

Note that you must add the initial drop of 10 m to the sum of the geometric series.

7. You can break the value 0.7 up into pieces as follows:

\[ 0.7 + 0.07 + 0.007 + \ldots \]

This series is a geometric series with an initial value of 0.7. The value of \( r \) is:

\[
\frac{t_2}{t_1} = \frac{0.07}{0.7} \\
= 0.1
\]

Now, since the 7 in the value 0.7 repeats an infinite number of times, you can use the sum to infinity of the geometric series to work out the ordinary fraction that 0.7 represents.

\[
S_\infty = \frac{0.7}{1-0.1} \\
= \frac{0.7}{0.9} \\
= \frac{7}{9}
\]

Summary

In Lesson 6, you learnt all about geometric sequences and series.

You learnt to define a geometric sequence by an initial value \( (a) \), and a common ratio, represented by the letter \( r \). The common ratio is a constant value that you multiply by each term in the sequence to produce the next term. Therefore, a geometric sequence takes the form:

\[ a + ar + ar^2 + \ldots \]
You also learnt how derive a general formula for the $n$th term in a geometric sequence, as well as the formula for the sum of the terms in a geometric series.

Finally, you learnt about a special type of geometric series, called a convergent geometric series. A series is convergent if $-1 < r < 1$.

The sum of a convergent geometric series tends towards a particular value as the number of terms in the series increases. However, sum will never actually reach that value. We express this mathematically by using the concept of a limit. This is a very useful concept, because it enables us to define a second concept, namely the sum to infinity of the series, which we define as:

$$S_n = \frac{a}{1-r}$$

![Self-assessment Questions 6](image)

Test your knowledge of this lesson by completing the self-assessment questions below. When you answer the questions, don’t look at the suggested answers that we give straight away. Look at them only after you’ve written your answers down, and then compare your answers with our answers.

1. The fifth term of a GS is 8, and the eighth term is 27. Find the first three terms of the sequence.

2. The length of any pipe of a certain organ is $\frac{7}{8}$ of the length of the previous pipe. If the longest pipe is 3 m in length, what is the length of the 20th pipe? Express your answer to the nearest centimetre.

3. 12; 16; $x$; . . . are the first three terms of a sequence. What is the value of $x$ in the sequence if the sequence is:
   (a) an arithmetic sequence?
   (b) a geometric sequence?

4. Which term of the sequence 1 728; 144; 12; . . . will be $\frac{1}{144}$?

5. The first, third and eleventh terms of a AS are also the first three terms of a GS. If the first term is 1, determine the GS.

6. Prove without using a calculator that $6 + 2\sqrt{5}$; 4; $6 - 2\sqrt{5}$ are three consecutive terms of a GS.

7. The sum of an infinite convergent GS is 24. However, if the same GS is taken, but only the infinite number of even-numbered terms (that is, $T_2 + T_4 + T_6 + \ldots$) are added, then the sum is 6. Find the first three terms of the GS.

8. All the terms of a convergent GS are positive. The sum of the first two terms is 5. The sum to infinity is 9. Find the first three terms of the GS.
Suggested answers to Self-assessment Questions 6

1. In order to find the first three terms of the sequence, you need to calculate the values of $a$ and $r$.

\[ t_5 = ar^4 \]
\[ 8 = ar^4 \]
\[ a = \frac{8}{r^4} \]

You also know that the eighth term is 27. Therefore:

\[ t_8 = ar^7 \]
\[ 27 = ar^7 \]

If you substitute the value of $a$ from the first equation into the second equation, then you have:

\[ 27 = \frac{8}{r^4} \times r^7 \]
\[ 27 = 8r^3 \]
\[ r^3 = \frac{27}{8} \]
\[ = \left(\frac{3}{2}\right)^3 \]

Therefore, $r$ is equal to $\frac{3}{2}$. Now you can work out the value of $a$:

\[ a = \frac{8}{\left(\frac{3}{2}\right)^4} \]
\[ = \frac{8}{81} \]
\[ = 8 \times \frac{16}{81} \]
\[ = \frac{128}{81} \]

Therefore, the first three terms are:

\[
\frac{128}{81}, \quad \frac{64}{27}, \quad \frac{32}{9}
\]
2. The lengths of the pipes in the organ form a geometric sequence. The first term is 3, and the common ratio is \( \frac{7}{8} \). Therefore the length of the 20th pipe is:

\[
t_{20} = ar^{n-1}
\]

\[
= 3 \times \left( \frac{7}{8} \right)^{19}
\]

\[
= 0.237
\]

Therefore, the length of the 20th pipe is 0.237 m, which is approximately 24 cm.

3. (a) If the sequence is an arithmetic sequence, then the sequence has an initial value of 12, and a common difference of 4. Therefore, \( x \) is:

\[
16 + 4 = 20
\]

(b) If the sequence is a geometric sequence, then the initial value is 12, and the common ratio is \( \frac{16}{12} \), or \( \frac{4}{3} \). Therefore, the value of \( x \) is:

\[
16 \times \frac{4}{3} = \frac{64}{3}
\]

\[
= 21 \frac{1}{3}
\]

4. The sequence is a geometric sequence with an initial value of 1728, and a common ratio of:

\[
r = \frac{12}{144}
\]

\[
= \frac{1}{12}
\]

Therefore, you need to solve for \( n \) in the following formula:

\[
t_n = 1728 \times \left(\frac{1}{12}\right)^{n-1}
\]

\[
\frac{1}{144} = 1728 \times \left(\frac{1}{12}\right)^{n-1}
\]

\[
\frac{1}{12^2} = 12^3 \times \left(\frac{1}{12}\right)^{n-1}
\]

\[
\frac{1}{12^5} = \left(\frac{1}{12}\right)^{n-1} \quad \text{(divide both sides by } 12^3 \text{)}
\]

\[
\left(\frac{1}{12}\right)^5 = \left(\frac{1}{12}\right)^{n-1}
\]
Therefore:

\[ n - 1 = 5 \]

\[ n = 6 \]

Therefore, the sixth term is \( \frac{1}{144} \).

5. You know that the first term in the arithmetic sequence is 1, which is also the value of \( a \). You also know that

\[ t_3 = a + 2d \]

\[ t_{11} = a + 10d \]

Therefore, in the geometric sequence:

\[ \frac{t_2}{t_1} = a + 2d \]

And:

\[ \frac{t_3}{t_2} = a + 10d \]

Since \( r \) is equal to both \( \frac{t_2}{t_1} \) and \( \frac{t_3}{t_2} \), and \( a \) is equal to 1, you have:

\[ \frac{a + 10d}{a + 2d} = \frac{a + 2d}{a} \]

\[ \frac{1 + 10d}{1 + 2d} = 1 + 2d \]

\[ 1 + 10d = (1 + 2d)^2 \]

\[ 4d^2 - 6d = 0 \]

Therefore, \( d = 0 \) or \( d = \frac{3}{2} \). This means that there are two possible geometric sequences, namely:

1; 1; 1; . . .

Or:

1; 4; 16; . . .
6. If the three values are three consecutive terms in a geometric sequence, then there will be a constant ratio between each successive term. The ratios are:

\[
\frac{t_2}{t_1} = \frac{4}{6 + 2\sqrt{5}}
\]

And:

\[
\frac{t_3}{t_2} = \frac{6 - 2\sqrt{5}}{4}
\]

For this to be a geometric sequence:

\[
\frac{4}{6 + 2\sqrt{5}} = \frac{6 - 2\sqrt{5}}{4}
\]

\[
1 = \frac{6 - 2\sqrt{5}}{4} \times \frac{6 + 2\sqrt{5}}{4} \quad \text{(multiply both sides by } \frac{6 + 2\sqrt{5}}{4} \text{)}
\]

\[
= \frac{36 + 12\sqrt{5} - 12\sqrt{5} - 20}{16}
\]

\[
= \frac{16}{16}
\]

\[
= 1
\]

Therefore, the three terms are part of a geometric sequence.

7. The sum to infinity of the series is:

\[
S_\infty = \frac{a}{1 - r}
\]

\[
24 = \frac{a}{1 - r}
\]

\[
a = 24(1 - r)
\]

The second series consists of the even-numbered terms of the first series, as follows:

\[
t_2 = ar
\]

\[
t_4 = ar^3
\]

\[
t_6 = ar^3
\]

Therefore, \(a\) is equal to \(ar\), and you can calculate the common ratio for this second series as follows:

\[
\frac{t_2}{t_1} = \frac{ar^3}{ar} = r^2
\]
Therefore, the sum to infinity for the second series is:

\[ S_\infty = \frac{a}{1-r} \]

\[ 6 = \frac{ar}{1-r^2} \]

\[ 6(1 - r^2) = ar \]

Now if you substitute the value for \( a \) from the first equation into this one, you have:

\[ 6(1 - r^2) = 24(1 - r)r \]

\[ 6 - 6r^2 = 24r - 24r^2 \]

\[ 18r^2 - 24r + 6 = 0 \]

\[ 3r^2 - 4r + 1 = 0 \]

(divide by 6)

\[ (3r - 1)(r - 1) = 0 \]

Therefore, \( r = \frac{1}{3} \) or 1. However, you know that the original geometric series is a convergent series, so \( r < 1 \). Therefore, \( r = \frac{1}{3} \). Now you can calculate \( a \) from the first equation:

\[ a = 24(1 - r) \]

\[ = 24(1 - \frac{1}{3}) \]

\[ = 24 \times \frac{2}{3} \]

\[ = 16 \]

Therefore, the first three terms of the original series are 16, \( \frac{16}{3} \) and \( \frac{16}{9} \).

8. The sum of the first two terms is 5. Therefore:

\[ a + ar = 5 \]

\[ a(1 + r) = 5 \]

The sum to infinity of the series is 9.

Therefore:

\[ \frac{a}{1-r} = 9 \]

\[ a = 9(1 - r) \]

If you substitute this value for \( a \) into the first equation, you have:
\[9(1 - r) (1 + r) = 5\]
\[9(1 - r^2) = 5\]
\[1 - r^2 = \frac{5}{9}\]
\[r^2 = \frac{4}{9}\]
\[r = \pm \frac{2}{3}\]

However, you know that all the terms in the series are positive, so \(r\) must be equal to \(\frac{2}{3}\). Now you can calculate the value of \(a\) as follows:

\[a = 9(1 - r)\]
\[= 9 \times \frac{1}{3}\]
\[= 3\]

Therefore, the first three terms of the series are 3, 2 and \(\frac{4}{3}\).

△ Check your competence

Now that you have worked through this lesson, please check that you can perform the tasks below:

- I can simplify expressions using the laws of exponents for rational exponents.
- I can add, subtract, multiply and divide simple surds.
- I can demonstrate an understanding of error margins.
- I can investigate number patterns, and hence:
  - make conjectures and generalisations; and
  - provide explanations and justifications, and attempt to prove conjectures.
- I can identify and solve problems involving number patterns, including, but not limited to arithmetic and geometric sequences and series.
- I can correctly interpret recursive formulae (for example, \(T_{n+1} = T_n + T_{n-1}\)).
- I can solve non-routine, unseen problems.

The next lesson

If you are sure that you understand the contents covered in this lesson, and have achieved your Assessment Standards, start Lesson 7.
LESSON 7: SIGMA NOTATION

Learning Outcomes for Lesson 7

After you have worked through Lesson 1, you should be able to do the following:

<table>
<thead>
<tr>
<th>LO</th>
<th>Assessment Standard</th>
</tr>
</thead>
<tbody>
<tr>
<td>LO 1</td>
<td>AS 11.1.2</td>
</tr>
<tr>
<td></td>
<td>(a) Simplify expressions using the laws of exponents for rational exponents.</td>
</tr>
<tr>
<td></td>
<td>(b) Add, subtract, multiply and divide simple surds.</td>
</tr>
<tr>
<td></td>
<td>(c) Demonstrate an understanding of error margins.</td>
</tr>
<tr>
<td></td>
<td>AS 11.1.6 Solve non-routine, unseen problems.</td>
</tr>
<tr>
<td></td>
<td>AS 12.1.3</td>
</tr>
<tr>
<td></td>
<td>(a) Identify and solve problems involving number patterns, including, but not limited to arithmetic and geometric sequences and series.</td>
</tr>
<tr>
<td></td>
<td>(b) Correctly interpret sigma notation.</td>
</tr>
<tr>
<td></td>
<td>(c) Prove and correctly select the formula for, and calculate the sum of a series. Correctly interpret recursive formulae (for example, $T_{n+1} = T_n + T_{n-1}$).</td>
</tr>
<tr>
<td></td>
<td>AS 12.1.6 Solve non-routine, unseen problems.</td>
</tr>
</tbody>
</table>

As before, we would like to help you to remain in complete control of your studies. So, once again, we provide a checkbox at the end of this lesson.

Introduction

So far, you have learnt about arithmetic and geometric sequences and series. In both cases, you learnt how to find a general formula for the $n$th term of a series, as well as a formula for the sum of the series.

In this lesson, we'll use your existing knowledge to introduce you to a new type of notation that's especially useful for working with a series of numbers. We call this notation sigma notation.

In this lesson, you'll learn:

- what sigma notation is;
- how to expand a series that is written in sigma notation;
- how to sum a series written in sigma notation; and
- how to write a series that is given in expanded form in sigma notation.
Sigma notation

Sigma notation offers a useful shorthand method to indicate the sum of a number of terms in a series. The symbol $\Sigma$ is the Greek capital letter $S$, and indicates sum. We use the symbol as follows:

$$\sum_{k=0}^{6} (3 + 4k)$$

In this example, we read the sigma notation as ‘the sum of the values of $(3 + 4k)$, with $k$ ranging from 0 to 6’.

The sigma notation refers to the entire expression, not just to the $\Sigma$ symbol.

The sigma notation is a short way to express a series. However, we can also write the series out in full. In this case, the full series in expanded form is:

$$3 + 7 + 11 + 15 + 19 + 23 + 27 = 105$$

Therefore:

$$\sum_{k=0}^{6} (3 + 4k) = 105$$

Table 10 shows how we arrived at each of the values in the series.

<table>
<thead>
<tr>
<th>$k$</th>
<th>$3 + 4k$</th>
<th>Term</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>3 + 4(0)</td>
<td>3</td>
</tr>
<tr>
<td>1</td>
<td>3 + 4(1)</td>
<td>7</td>
</tr>
<tr>
<td>2</td>
<td>3 + 4(2)</td>
<td>11</td>
</tr>
<tr>
<td>3</td>
<td>3 + 4(3)</td>
<td>15</td>
</tr>
<tr>
<td>4</td>
<td>3 + 4(4)</td>
<td>19</td>
</tr>
<tr>
<td>5</td>
<td>3 + 4(5)</td>
<td>23</td>
</tr>
<tr>
<td>6</td>
<td>3 + 4(6)</td>
<td>27</td>
</tr>
</tbody>
</table>

Can you see how we have expanded the sigma notation? The expression in brackets contains a variable. That variable also appears just below the sigma symbol, and shows the starting value of a range of numbers. In this case, the first value is 0. Then, just above the sigma symbol is another number. This is the end of the range of the numbers. In our example, the end value is 6.

So, the sigma notation means that we write out the term in the brackets, once for each value of $k$ in the range specified.

Earlier, we stated that we read the sigma notation as ‘the sum of the values of $(3 + 4k)$, with $k$ ranging from 0 to 6’. You can see now how the sigma notation neatly summarises that entire phrase.
The sigma notation expression represents the actual sum of the series, not the series itself.

For example:

\[ \sum_{k=1}^{5} k = 1 + 2 + 3 + 4 + 5 \]
\[ = 15 \]

\[ \sum_{r=0}^{3} 2^r = 2^0 + 2^1 + 2^2 + 2^3 \]
\[ = 15 \]

Now, would you say that the following expression is true, or false?

\[ \sum_{k=1}^{5} k = \sum_{r=0}^{3} 2^r \]

The expression is true, because the sigma notation represents the sum of each series. Therefore, each sigma notation expression represents the value 15, and so the two are equal.

Let’s look at a few more examples of how to write an expression given in sigma notation in expanded form.

EXAMPLE

1. \[ \sum_{k=0}^{5} 3k^2 = 3(0^2) + 3(1^2) + 3(2^2) + 3(3^2) + 3(4^2) + 3(5^2) \]
\[ = 0 + 3 + 12 + 27 + 48 + 75 \]
\[ = 165 \]

2. \[ \sum_{i=0}^{7} 2(-1)^i = 2(-1)^0 + 2(-1)^1 + 2(-1)^2 + 2(-1)^3 + \ldots + 2(-1)^7 \]
\[ = 2 - 2 + 2 - 2 + 2 - 2 + 2 - 2 \]
\[ = 0 \]

3. \[ \sum_{n=1}^{4} \frac{n}{n+1} = \frac{1}{1+1} + \frac{2}{2+1} + \frac{3}{3+1} + \frac{4}{4+1} \]
\[ = \frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} \]
\[ = 2.716 \]  
   (Use your calculator)
4. \[ \sum_{k=4}^{7} (3x + k) = (3x + 4) + (3x + 5) + (3x + 6) + (3x + 7) \]
   \[ = 4 \times 3x + (4 + 5 + 6 + 7) \]
   \[ = 12x + 22 \]

The number of terms in the fully expanded series is:

\[ b - a + 1 \]

where \( a \) is the start of the range of value in the series, and \( b \) is the end of the range of values.

For example, \[ \sum_{k=1}^{9} k^2 \] has nine terms in the series to add \((9 - 1 + 1)\), while \[ \sum_{m=3}^{10} 2^{-m} \] has eight terms to add \((10 - 3 + 1)\).

**Evaluate a sigma notation expression**

A good way to start when you have to evaluate a sigma notation expression is to write down the first few terms in the series, and to evaluate them individually. If there are only a few terms to add, it is probably easier simply to add them.

However, if there are many terms to add, it is advisable to look for a pattern in the series. If the series is either an AS or a GS, use the appropriate sum formula.

Let’s look an example of how to do this.

---

**EXAMPLE**

Evaluate the following:

1. \[ \sum_{r=1}^{4} \left(2^r + r^2\right) \]

2. \[ \sum_{k=4}^{23} (3k - 1) \]

**Answer**

1. \[ \sum_{r=1}^{4} \left(2^r + r^2\right) = (2 + 1) + (4 + 4) + (8 + 9) + (16 + 16) \]
   \[ = 3 + 8 + 17 + 32 \]
   \[ = 60 \]

2. \[ \sum_{k=4}^{23} (3k - 1) = 11 + 14 + 17 + \ldots \text{ (to 20 terms)} \]

This series is an AS. Therefore, you can use the following formula to evaluate the sum of the series:

\[ S_n = \frac{n}{2} (2a + (n - 1)d) \]
In this case, the first term is 11, and the common difference is 3. There are 20 terms in the series. Therefore:

\[
S_{20} = \frac{20}{2}(2[11] + (20 - 1)3) \\
= 10(22 + 57) \\
= 790
\]

Therefore, \( \sum_{k=4}^{23} (3k - 1) = 790 \).

You must also be able to express a series written in expanded form using sigma notation.

**Express a series in sigma notation**

In order to express a series in sigma notation, you need to use your knowledge of how to find the general formula for the \( n \)th term in the series. Specifically, you need to work out:

- what the general term of the series is; and
- how many terms must be added together.

Once you have worked out the general term for the series, this forms the expression that you place in the brackets in front of the sigma symbol. The number of terms in the series dictates the start and end values of the range that you insert above and below the sigma symbol.

Let’s work through a few examples to see how this works.

**EXAMPLE**

Write the following series in sigma notation:

1. \( 1 + 3 + 5 + \ldots \) (to 12 terms)
2. \( 2 + 5 + 8 + \ldots + 44 \)
3. \( 27 - 9 + 3 - \ldots - \frac{1}{81} \)
4. \( 1 \times 2 + 2 \times 3 + 3 \times 4 + \ldots \) (to \( n \) terms)

**Answer**

1. This series forms an AS with general term \( a + (n - 1)d \). In this case:

\[
\begin{align*}
  a &= 1 \\
  d &= 2 \\
  n &= 12
\end{align*}
\]
Therefore:
\[ a + (n - 1)d = 1 + (n - 1) \times 2 \]
\[ = 2n - 1 \]

Therefore, in sigma notation, the sum of this series is:
\[ \sum_{n=1}^{12} (2n - 1) \]

2. This series forms an AS, but you don’t yet know how many terms there are in the series. All you know is that the last term is 44. You do know that:
\[ a = 2 \]
\[ d = 3 \]
\[ t_n = 44 \]

Therefore:
\[ 44 = a + (n - 1)d \]
\[ = 2 + (n - 1) \times 3 \]
\[ = 2 + 3n - 3 \]
\[ 45 = 3n \]
\[ n = 15 \]

The general term for the series is:
\[ a + (n - 1)d = 2 + (n - 1) \times 3 \]
\[ = 3n - 1 \]

Therefore, in sigma notation, the sum of this series is:
\[ \sum_{n=1}^{15} (3n - 1) \]

3. This series forms a GS. You know that:
\[ a = 27 \]
\[ r = -3 \]
\[ t_n = -\frac{1}{81} \]
Therefore:

\[-\frac{1}{81} = ar^{n-1}\]

\[-\frac{1}{3^7} = 27 \times \left(-\frac{1}{3}\right)^{n-1}\]

\[\left(-\frac{1}{3}\right)^7 = \left(-\frac{1}{3}\right)^{n-1}\]

\[n - 1 = 7\]

\[n = 8\]

The general term for the series is:

\[ar^{n-1} = 27\left(-\frac{1}{3}\right)^{n-1}\]

Therefore, in sigma notation, the sum of this series is:

\[\sum_{n=1}^{8} 27\left(-\frac{1}{3}\right)^{n-1}\]

4. This is a more complicated example, because the series is not obviously an AS or a GS. However, if you split each term in the series into two parts, then the result is two separate AS. Then you can work with each separately, before recombining them at the end to form the original series.

The original series is:

\[1 \times 2 + 2 \times 3 + 3 \times 4 + \ldots \text{ (to } n \text{ terms)}\]

Each term consists of factors separated by a multiplication sign. First, consider the series formed by the first factor in each term. These numbers form the following AS, with \(a = 1\), and \(d = 1\):

\[1 + 2 + 3 + \ldots \text{ (to } n \text{ terms)}\]

The general term for this series is

\[a + (k - 1)d = 1 + (k - 1)\]

\[= k\]

Now consider the series formed by the numbers in the second factor for each term. These numbers form the following AS, with \(a = 2\), and \(d = 1\):

\[2 + 3 + 4 + \ldots \text{ (to } n \text{ terms)}\]

The general term for this series is

\[a + (k - 1)d = 2 + (k - 1)\]

\[= k + 1\]
Now you can rejoin the two separate series to form the original series. In other words, each term in the original series has a general term of \(k (k + 1)\). Therefore, in sigma notation, the sum of this series is:

\[
\sum_{k=1}^{n} k (k + 1)
\]

**NB** You should use the symbol \(k\) rather than \(n\) in the general term. The reason is that you need to indicate the \(n\) terms in the series, which must appear above the sigma symbol. It would be very confusing if \(n\) appeared both above the sigma symbol and in the general term of the sigma notation.

Now work through the following activity to practise what you have learnt about sigma notation.

### Activity 12

1. Find the value of:

   (a) \(\sum_{r=0}^{3} \frac{2r}{2^r}\)

   (b) \(\sum_{k=1}^{80} (5k - 2)\)

   (c) \(\sum_{n=0}^{6} \frac{1}{2^n}\)

2. Find the value of \(n\) if \(\sum_{r=1}^{n} 2(6 - r) = 24\).

3. Find the value of \(m\) if \(\sum_{k=1}^{m} (-16)\left(\frac{1}{2}\right)^k = 5 \frac{1}{2}\).

4. If \(\sum_{m=2}^{16} \log_k m^{m-1} = 120(\log k)^2\), calculate the value of \(k\).

5. Calculate the value of \(n\) if \(\sum_{p=1}^{18} (7p - 22) - \sum_{k=1}^{n} \frac{3^k}{2^k} = 52 \frac{7}{8}\).

**Answers**

1. (a) \(\sum_{r=0}^{3} \frac{2r}{2^r} = \frac{2(0)}{2^0} + \frac{2(1)}{2^1} + \frac{2(2)}{2^2} + \frac{2(3)}{2^3}\)

   \[= 0 + 1 + \frac{3}{4}\]

   \[= 2 \frac{3}{4}\]
(b) \[ \sum_{k=1}^{80} (5k - 2) = (5(1) - 2) + (5(2) - 2) + (5(3) - 2) + \ldots + (5(80) - 2) \]
\[ = 3 + 8 + 13 + \ldots + 398 \]
This is an AS. Therefore, the sum of the series is:
\[ S_n = \frac{n}{2} (t_1 + t_n) \]
\[ = \frac{80}{2} (3 + 398) \]
\[ = 16040 \]

(c) \[ \sum_{n=0}^{6} \frac{1}{2^n} = 1 + \frac{1}{2} + \frac{1}{4} + \ldots \text{ (to 7 terms)} \]
This is a GS, with \( a = 1 \), \( r = \frac{1}{2} \), and \( n = 7 \). Therefore:
\[ S_n = \frac{a(1 - r^n)}{1 - r} \]
\[ = \frac{1 - \left(\frac{1}{2}\right)^7}{1 - \frac{1}{2}} \]
\[ = \frac{127}{64} \]
\[ = 1 \frac{63}{64} \]

2. First, write out the first few terms of the series:
\[ (2(6 - 1)) + (2(6 - 2)) + (2(6 - 3)) + \ldots = 24 \]
\[ 10 + 8 + 6 + \ldots = 24 \]
This is an AS, with \( a = 10 \) and \( d = -2 \). You also know that \( S_n = 24 \). Therefore:
\[ \frac{n}{2} (2[10] + (n - 1) \times (-2)) = 24 \]
\[ 10n - n^2 + n = 24 \]
\[ n^2 - 11n + 24 = 0 \]
\[ (n - 3)(n - 8) = 0 \]
Therefore, \( n = 3 \), or \( n = 8 \).

3. First, write out the first few terms of the series:
\[ \sum_{k=1}^{m} (-16) \left(\frac{1}{2}\right)^k = (-16) \left(\frac{1}{2}\right)^1 + (-16) \left(\frac{1}{2}\right)^2 + (-16) \left(\frac{1}{2}\right)^3 + \ldots \]
\[ = 8 - 4 + 2 - \ldots \]
This is a GS, with \( a = 8 \) and \( r = -2 \). You also know that \( S_m = 5 \frac{1}{2} \). Therefore:

\[
S_m = \frac{8\left(1 - \left(-\frac{1}{2}\right)^m\right)}{1 - \left(-\frac{1}{2}\right)}
\]

\[
\frac{11}{2} = \frac{8 - 8\left(-\frac{1}{2}\right)^m}{\frac{3}{2}}
\]

\[
\frac{11}{2} \times \frac{3}{2} = 8 - 8\left(-\frac{1}{2}\right)^m
\]

\[
\frac{33}{4} - 8 = -8\left(-\frac{1}{2}\right)^m
\]

\[
\frac{1}{4} \times -\frac{1}{8} = \left(-\frac{1}{2}\right)^m \quad \text{(Divide both sides by -8)}
\]

\[
\left(-\frac{1}{2}\right)^5 = \left(-\frac{1}{2}\right)^m
\]

Therefore, \( m = 5 \).

4. \[\sum_{m=2}^{16} \log k^{m-1} = \log k + \log k^2 + \log k^3 + \ldots + \log k^{15}\]

This series is an AS:

\[
a = \log k
\]

\[
d = \log k
\]

\[
n = 16 - 2 + 1
\]

\[
= 15
\]

Therefore, you can use the following formula to evaluate the sum of the series:

\[
S_n = \frac{n}{2} (2a + (n - 1)d)
\]

Therefore:

\[
\frac{15}{2} (2 \log k + (15 - 1) \log k) = 120 (\log k)^2
\]

\[
\frac{15}{2} (16 \log k) = 120 (\log k)^2
\]

\[
120 \log k = 120 (\log k)^2
\]

\[
(\log k)^2 - \log k = 0
\]

\[
\log k (\log k - 1) = 0
\]
Therefore:

\[ \log k = 0, \text{ and } k = 1 \]

Or:

\[ \log k = 1, \text{ and } k = 10 \]

5. First, write out each series, and determine if it’s an AS or GS.

\[
\sum_{p=1}^{18} (7p - 22) = (7(1) - 22) + (7(2) - 22) + (7(3) - 22) + \ldots + (7(18) - 22)
\]

\[ = -15 - 8 - 1 \ldots + 104 \]

This series is an AS:

\[
a = -15 \\
d = 7 \\
n = 18
\]

Therefore, you can use the following formula to evaluate the sum of the series:

\[ S_n = \frac{n}{2} (2a + (n - 1)d) \]

\[ = \frac{18}{2} (2(-15) + (18 - 1)7) \]

\[ = \frac{18}{2} (89) \]

\[ = 801 \]

Now, calculate the sum of the second series in the expression:

\[
\sum_{k=1}^{n} 24 \left( \frac{3}{2} \right)^k = 24 \left( \frac{3}{2} \right)^1 + 24 \left( \frac{3}{2} \right)^2 + 24 \left( \frac{3}{2} \right)^3 + \ldots
\]

This is a GS:

\[
a = 24 \left( \frac{3}{2} \right) \\
r = \frac{3}{2}
\]

Therefore, you can use the following formula to evaluate the sum of the series:

\[ S_n = a \frac{1 - r^n}{1 - r} \]

\[ = 24 \left( \frac{3}{2} \right) \frac{1 - \left( \frac{3}{2} \right)^n}{1 - \frac{3}{2}} \]

\[ = \frac{36 \left( 1 - \left( \frac{3}{2} \right)^n \right)}{-\frac{1}{2}} \]
Now, rewrite the equation in the question using these two formulas for the sum of the each of the two series.

Then, solve for \( n \).

\[
801 - \frac{36(1-(\frac{3}{2})^n)}{-\frac{1}{2}} = 52 \frac{7}{8}
\]

\[
801 + 72 - 72\left(\frac{3}{2}\right)^n = 423 \frac{3}{8}
\]

\[
72\left(\frac{3}{2}\right)^n = \frac{6561}{8}
\]

\[
\left(\frac{3}{2}\right)^n = \frac{6561}{576}
\]

(divide both sides by 72)

\[
= \frac{729}{64}
\]

\[
= \left(\frac{3}{2}\right)^6
\]

\[n = 6\]

**Summary**

In this lesson, you learnt about a useful shorthand method for expressing the sum of a series, called sigma notation.

An example of sigma notation is:

\[
\sum_{k=1}^{5} 2k
\]

This sigma notation expression states that you must replace the letter \( k \) by the values 1 to 5, to produce the successive terms in a series. The expression represents the resultant sum of the series, and not the terms in the series.

You also learnt how to express a series written in expanded form in sigma notation. In order to do so, you need to know:

- what the general term of the series is; and
- how many terms must be added together.

In the general form of the sigma notation expression, \( \sum_{k=a}^{b} \left( . . . \right) \), the number of terms in the series is \( b - a + 1 \).

If the terms in the series form an arithmetic or geometric series, then you can use the appropriate sum formula to find the sum of the series.
Test your knowledge of this lesson by completing the self-assessment questions below. When you answer the questions, don’t look at the suggested answers that we give straight away. Look at them only after you’ve written your answers down, and then compare your answers with our answers.

1. Show that the sequence \( \log 4; \log 20; \log 100; \log 500; \log 2 \, 500 \) is an AS.

2. Calculate the sum of the first 19 terms of an AS of which the fourth term is 14, and the seventh term is 23.

3. How many terms in the sequence 11; 18; 25; \ldots must be added to obtain an answer of 4 301?

4. Which term of the sequence 92; 81; 70; \ldots is equal to \(-150\)?

5. The first three terms of a GS are \( k - 2, 2k - 6 \) and \( 4k - 8 \).
   (a) Solve for \( k \).
   (b) Find the sum of the first six terms of the sequence.

6. Evaluate \( \sum_{k=4}^{14} 3 \times 2^{5-k} \) to three decimal places.

7. Find the common ratio of a GS of which the sixth term is 96 and the first term is 3.

8. A stack of bricks has one brick in the top layer, two in the second layer, three in the third layer, and so on. Determine how many layers are in the stack if there are 190 bricks altogether.

9. The sum of the first two terms of a GS is \(-4\). The sum of the fourth and fifth terms of the same sequence is 108. Find the third term.

10. The sum of a series is \( 2n^2 \). Find a formula in terms of \( n \) for \( t_n \), the \( n \)th term in the series.

11. What is the largest possible value for \( n \) if \( \sum_{r=1}^{n} (2r + 5) < 250 \)?

12. For each of the following infinite series, say whether it converges or diverges, and if it converges, find the sum to infinity:
   (a) \( \sum_{n=0}^{\infty} \left( \frac{-7}{8} \right)^n \)
   
   (b) \( \sum_{n=0}^{\infty} (-1)^n \)

   (c) \( \sum_{n=0}^{\infty} 10^{-6} (1,1)^n \)
1. Calculate the difference between each of the successive terms. If the difference is constant, then the sequence is an arithmetic sequence.

\[ t_2 - t_1 = \log 20 - \log 4 \]
\[ = \log \frac{20}{4} \]
\[ = \log 5 \]
\[ t_3 - t_2 = \log 100 - \log 20 \]
\[ = \log \frac{100}{20} \]
\[ = \log 5 \]
\[ t_4 - t_3 = \log 500 - \log 100 \]
\[ = \log \frac{500}{100} \]
\[ = \log 5 \]
\[ t_5 - t_4 = \log 2500 - \log 500 \]
\[ = \log \frac{2500}{500} \]
\[ = \log 5 \]

The difference between the terms is constant. Therefore, the sequence is an arithmetic sequence.

2. The formula for the sum of the first 19 terms is:

\[ S_{19} = \frac{n}{2} (2a + (n - 1)d) \]
\[ = \frac{19}{2} (2a + 18d) \]
\[ = 19a + 171d \]

As you can see, you need to know the values of \( a \) and \( d \) to be able to work out the sum of the first 19 terms. You can work out these values by using the other information given to you in the question.

You know that the 4th term of the sequence is 14. Therefore:

\[ t_4 = a + (n - 1)d \]
\[ 14 = a + 3d \]
\[ a = 14 - 3d \]
You also know that the 7th term of the sequence is 23. Therefore:

\[ t_7 = a + (n - 1)d \]

\[ 23 = a + 6d \]

Now you can substitute the value of \( a \) from the first equation into the second equation, which gives you:

\[ 23 = 14 - 3d + 6d \]

\[ 9 = 3d \]

\[ d = 3 \]

Now you can substitute this value for \( d \) back into the first equation, and work out the value of \( a \), as follows:

\[ a = 14 - 3d \]

\[ = 14 - 9 \]

\[ = 5 \]

Now you can calculate the sum of the first 19 terms:

\[ S_{19} = 19a + 171d \]

\[ = 19(5) + 171(3) \]

\[ = 95 + 513 \]

\[ = 608 \]

3. The sequence is an AS, with \( a = 11 \) and \( d = 7 \). You also know that \( S_n = 4301 \). You need to solve for \( n \) in the following formula:

\[ \frac{n}{2} (2a + (n - 1)d) = 4301 \]

\[ n(22 + 7n - 7) = 8602 \]  

(multiply both sides by 2)

\[ 15n + 7n^2 = 8602 \]

\[ 7n^2 + 15n - 8602 = 0 \]

\[ n = \frac{-15 \pm \sqrt{(15)^2 - 4(7)(-8602)}}{2(7)} \]

Therefore, the two possible solutions to the equation are \( n = 34 \) and \( n = -36,14 \). Since \( n \) must be greater than 0, \( n = 34 \). Therefore, the sum of the first 34 terms is equal to 4301.

4. The first term of the sequence is 92 (the value of \( a \)). The common difference for the sequence is \( -11 \) (the value of \( d \)). Therefore, you need to solve for \( n \) in the general formula for the sequence.
Since $t_n = -150$:

\[-150 = a + (n - 1)d\]

\[= 92 + (n - 1)(-11)\]

\[= 103 -11n\]

\[11n = 253\]

\[n = 23\]

Therefore, the 23rd term is equal to $-150$.

5. (a) There is a common ratio between each successive term. Therefore:

\[\frac{t_2}{t_1} = \frac{2k - 6}{k - 2}\]

\[\frac{t_3}{t_2} = \frac{4k - 8}{2k - 6}\]

The ratio is constant, which means that:

\[\frac{2k - 6}{k - 2} = \frac{4k - 8}{2k - 6}\]

\[(2k - 6)^2 = (4k - 8)(k - 2)\]

\[4k^2 - 24k + 36 = 4k^2 - 16k + 16\]

\[20 = 8k\]

\[k = \frac{5}{2}\]

Therefore, the sequence is $\frac{1}{2}, -1; 2; \ldots$, with:

\[r = -2\]

\[a = \frac{1}{2}\]

(b) The formula for the sum of the first six terms is:

\[S_6 = \frac{a(1 - r^6)}{1 - r}\]

\[= \frac{\frac{1}{2}(1 - (-2)^6)}{1 - (-2)}\]

\[= \frac{\frac{1}{2}(1 - 64)}{3}\]
6. First, write out the series:

\[
\sum_{k=4}^{14} 3 \times 2^{5-k} = 3 \times 2^1 + 3 \times 2^0 + 3 \times 2^{-1} + 3 \times 2^{-2} + \ldots \text{ (to 11 terms)}
\]

\[
= 6 + 3 \times \frac{3}{2} + \frac{3}{4} + \ldots
\]

This series is a GS, with:

\[
r = \frac{1}{2}
\]

\[
a = 6
\]

\[
n = 11
\]

Therefore:

\[
S_{11} = \frac{6 \left[1 - \left(\frac{1}{2}\right)^{11}\right]}{1 - \frac{1}{2}}
\]

\[
= \frac{6 \left[1 - \left(\frac{1}{2}\right)^{11}\right]}{1 - \frac{1}{2}}
\]

\[
= 11,994
\]

7. You know the following:

\[
t_6 = ar^5
\]

\[
96 = 3r^5
\]

\[
r^5 = 32
\]

\[
r^5 = 2^5
\]

Therefore, \(r\) is equal to 2.

8. The number of bricks used is 1 + 2 + 3 + \ldots = 190. Therefore, this is an arithmetic series, with \(a = 1\), and \(d = 1\). You also know that \(S_n = 190\). You need to solve for \(n\) in the following formula:

\[
\frac{n}{2} (2(1) + (n - 1)1) = 190
\]

\[
r(2 + n - 1) = 380 \quad \text{(multiply both sides by 2)}
\]

\[
n + n^2 = 380
\]

\[
n^2 + n - 380 = 0
\]

\[
(n + 20)(n - 19) = 0
\]
Therefore, the two possible solutions to the equation are \( n = -20 \) and \( n = 19 \). Since \( n \) must be greater than 0, \( n = 19 \).

9. The sum of the first two terms is:

\[
a + ar = -4
\]

\[
a(1 + r) = -4
\]

The sum of terms four and five is:

\[
ar^3 + ar^4 = 108
\]

\[
r^3a(1+r) = 108
\]

Substitute the value of \( a(1 + r) \) into the second equation:

\[
-4 r^3 = 108
\]

\[
r^3 = -27
\]

\[
= (-3)^3
\]

Therefore, \( r \) is equal to \(-3\). Therefore, the third term is:

\[
t_3 = ar^2
\]

\[
= 2 \times (-3)^2
\]

\[
= 18
\]

10. Calculate the value of each term, and identify the type of the series. After that, you can use the formula for the \( n \)th term for that series to calculate \( t_n \). The first term, \( t_1 \), is also equal to \( S_1 \).

Therefore:

\[
t_1 = 2 \times (1)^2
\]

\[
= 2
\]

\[
t_2 = S_2 - S_1
\]

\[
= 2(2)^2 - 2
\]

\[
= 6
\]

\[
t_3 = S_3 - S_2
\]

\[
= 2(3)^2 - 8
\]

\[
= 18 - 8
\]

\[
= 10
\]
Therefore, the series is \( 2 + 6 + 10 + \ldots \) (to \( n \) terms). This is an AS, with:

\[
a = 2 \\
d = 4
\]

Therefore, the general term for the series is:

\[
t_n = 2 + (n - 1) \times 4 \\
= 4n - 2
\]

11. First, write out the series:

\[
\sum_{r=1}^{n} (2r + 5) = (2 \times 1 + 5) + (2 \times 2 + 5) + (2 \times 3 + 5) + \ldots \\
= 7 + 9 + 11 + \ldots
\]

The series is an AS, with:

\[
a = 7 \\
d = 2
\]

The sum of the series must be less than 250:

\[
\frac{n}{2} \left[ 2 \times 7 + (n - 1) \times 2 \right] < 250 \\
n^2 + 6n - 250 < 0
\]

Let \( n^2 + 6n - 250 = 0 \). Then:

\[
n = \frac{-6 \pm \sqrt{36 + 4(250)}}{2}
\]

The two possible values of \( n \) are 13,1 and \(-19,1\). In other words:

\[
(n - 13,1)(n + 19,1) < 0
\]

Therefore:

\[
n - 13,1 < 0 \\
n < 13,1
\]

And:

\[
n + 19,1 < 0 \\
n > -19,1
\]

Therefore, the largest possible value of \( n \) is 13.
12. If $-1 < r < 1$, then the series converges, otherwise it diverges.

(a) Write out the series:

$$
\sum_{n=0}^{\infty} \left( \frac{-7}{8} \right)^n = \left( \frac{-7}{8} \right)^0 + \left( \frac{-7}{8} \right)^1 + \left( \frac{-7}{8} \right)^2 + \ldots
$$

The value of $r$ in this series is $-\frac{7}{8}$, and $a$ is equal to 1. Therefore, the series converges. The sum to infinity is:

$$
S_n = \frac{1}{1 - \left(-\frac{7}{8}\right)}
$$

$$
= \frac{1}{\frac{15}{8}}
$$

$$
= \frac{8}{15}
$$

$$
= 0.53
$$

(b) Write out the series:

$$
\sum_{n=0}^{\infty} (-1)^n = (-1)^0 + (-1)^1 + (-1)^2 + \ldots
$$

The value of $r$ in this series is $-1$. Therefore, the series diverges.

(c) Write out the series:

$$
\sum_{n=0}^{\infty} 10^{-6}(1,1)^n = 10^{-6}(1,1)^0 + 10^{-6}(1,1)^1 + 10^{-6}(1,1)^2 + \ldots
$$

The value of $r$ in this series is $1,1$. Therefore, the series diverges.
Check your competence

Now that you have worked through this lesson, please check that you can perform the tasks below:

- I can simplify expressions using the laws of exponents for rational exponents.
- I can add, subtract, multiply and divide simple surds.
- I can demonstrate an understanding of error margins.
- I can identify and solve problems involving number patterns, including, but not limited to arithmetic and geometric sequences and series.
- I can correctly interpret sigma notation.
- I can prove and correctly select the formula for, and calculate the sum of a series.
- I can correctly interpret recursive formulae (for example, $T_{n+1} = T_n + T_{n-1}$).
- I can solve non-routine, unseen problems.

The next lesson

If you are sure that you understand the contents covered in this lesson, and have achieved your Assessment Standards, start Lesson 8.
# LESSON 8: COMPOUND INCREASE AND DECREASE

## Learning Outcomes for Lesson 8

After you have worked through Lesson 8, you should be able to do the following:

<table>
<thead>
<tr>
<th>LO</th>
<th>Assessment Standards</th>
</tr>
</thead>
<tbody>
<tr>
<td>LO 1</td>
<td>Use simple and compound decay formulae ( A = P (1 - ni) ) and ( A = P (1 - i)^n ) to solve problems (including, straight line depreciation and depreciation on a reducing balance).</td>
</tr>
<tr>
<td></td>
<td>AS 11.1.5 Demonstrate an understanding of different periods of compounding growth and decay (including, effective compounding growth and decay and including effective and nominal interest rates).</td>
</tr>
<tr>
<td></td>
<td>AS 11.1.6 Solve non-routine, unseen problems.</td>
</tr>
<tr>
<td></td>
<td>AS 12.1.4 Calculate the value of ( n ) in the formula ( A = P (1 \pm i)^n ).</td>
</tr>
<tr>
<td></td>
<td>AS 12.1.6 Solve non-routine, unseen problems.</td>
</tr>
</tbody>
</table>

We would like to help you to remain in complete control of your studies. So we give you an opportunity to check your competence at the end of this lesson.

## Introduction

Lesson 8 covers *compound increase and decrease*.

As you’ll see, compound increase and decrease are examples of geometric sequences. Therefore, in this lesson, you’ll begin to apply the knowledge that you have gained from the first seven lessons in the unit.

In the first part of the lesson, you’ll learn about compound increase, including how it applies to:

- interest rates;
- appreciation; and
- population growth.

In the second part of the lesson, we'll explain the concept of compound decrease. Here, you'll learn about:

- depreciation;
- radioactive decay; and
- inflation.
We've selected interest rates as the primary focus of the lesson, because they play such a large role in our everyday life.

Therefore, in this lesson, you'll learn about:

• simple interest and compound interest; and
• nominal and effective interest rates.

In the next lesson, you'll learn more about how to apply your knowledge of interest rates to various real-life situations.

**Interest rates**

When we invest money, we want it to become more. It becomes more by 'earning' interest.

*Interest* refers to a charge made for the use of someone else's money.

When we lend out money, we want more back than what we lent out and we achieve this by charging interest.

When we have borrowed money, we pay back more than what we borrowed and this happens because we pay interest. The financial institution calculates interest as a percentage of the amount originally borrowed or invested.

We'll begin our discussion on interest rates by looking at *simple interest*.

**Simple interest** ($I_S$)

A basic concern for financial managers is to determine the future value of current investments. The only reason that you invest money is so that it will be worth more in the future.

For example, suppose that someone offers you the choice of receiving R100 now, or R100 a year from now. Which option is worth more? In this case, the option of receiving the money now is always worth more, because you could invest the R100 now, receive interest on it for a year, and so have more than R100 a year from now. This illustrates the *time value of money*. In other words, the same amount of money is worth more (has a higher value) now than the same amount at some time in the future.

The option of investing money leads to a concept known as the time value of money.

*The time value of money* refers to the fact that the option of receiving R1 today is more valuable than receiving R1 at some time in the future.

So, what is *simple interest*?

*Simple interest* is interest earned only on the original amount invested.

To calculate simple interest, you need to multiply the original amount ($A$) by the interest rate ($i$), and the number of years ($n$) of the loan or the investment.
Mathematically, you calculate simple interest ($IS$) using the following formula:

$$IS = A \times n \times i$$

In the formula for simple interest, the subscript $S$ in $IS$ indicates that we are using simple interest in this calculation.

Consider the following example to help you to understand how to use this formula.

**EXAMPLE**

What is the value of the interest on an investment of R9 000, after twelve years, at 14% simple interest?

**Answer**

Remember, 14% is equal to $\frac{14}{100}$ or 0,14.

When you look at the question, you should be able to identify the following variables:

- $A = $R9 000
- $i = 14\%$ or 0,14
- $n = 12$ years

Substituting these quantities into the equation gives us:

$$IS = A \times n \times i$$

$$= 9 000 \times 12 \times 0,14$$

$$= 15 120$$

Therefore, the simple interest earned after twelve years is R15 120.

To find out what the investment will be worth after twelve years, we just add the interest that the investment has earned to the original amount invested.

Therefore, after twelve years the whole investment will be worth:

$$R9 000 + R15 120 = R24 120$$

Let’s see how we can develop a formula for calculating the amount an investment or a loan is worth without having to calculate the interest first and then adding it to the original amount.

From the above example, to find what the investment was worth after the 12 years, we added the original amount invested and the interests earned. As a formula, we obtain the following:

$$FS = A + IS$$
Where \( F_S \) = the sum of the original amount and the interests (Final Amount),
\[ A = \text{original amount invested} \]
\[ I_S = \text{interests earned}. \]

But remember that \( I_S = A \times n \times i \).

So if we substitute \( I_S = A \times n \times i \) in the formula \( F_S = A + I_S \), we obtain the following:
\[ F_S = A + (A \times n \times i) \]

Simplifying the formula by taking a common factor of \( A \), we obtain:
\[ F_S = A(1 + ni) \]

Now work carefully through the following activity to practise what you've learnt about simple interest.

### Activity 13

You inherited R15 000 from a distant relative. You wish to invest the whole amount for 10 years. You then choose to invest your money at the Central Bank, which pays you 12% simple interest per year. Use the formula for simple interest to calculate how much money you will have in your account after 10 years.

**Answer**

\[
F_S = A(1 + ni) \\
= 15 000(1 + 10 \times 0.12) \\
= 15 000(2.2) \\
= R33 000
\]

Therefore, after 10 years, you will have R33 000 in your account.

Let’s work through another example.

### EXAMPLE

Mrs. Sickle wants to invest an amount of R14 000 for the education of her daughter, who will matriculate in nine years’ time.

1. At what interest rate (simple interest) should she invest the money if the university fees will be R50 000 by the time her daughter matriculates? Round off your answer to the nearest percentage.

2. If she decides to take out R1 000 to buy a bicycle for her son, and then invests the remaining amount at a simple interest rate of 33% for seven years, will she still be able to afford to pay the fees for her daughter?
Answer

1. From the information given, we have the following:

\[ A = R14\,000 \]
\[ n = 9 \]
\[ i = ? \]
\[ F_S = R50\,000 \]

Therefore, we need to find the value of \( i \) in the following equation:

\[ F_S = A \times (1 + ni) \]

This gives us:

\[
\frac{50\,000}{14\,000} = 1 + 9i
\]

\[
9i = \frac{50}{14} - 1
\]

\[
9i = \frac{36}{14}
\]

\[
i = \frac{36}{14} \times 9
\]

\[
i = 0,285714 \quad \text{(Don’t round off yet)}
\]

\[
i = 28,5714 \quad \text{(which is approximately 29%)}
\]

Therefore, Mrs. Sickle should invest the money at a simple interest rate of 29%. Note that we need to round the final percentage up, because if we rounded it down, then Mrs. Sickle would not reach her goal of R50 000. The final value of the investment would be slightly less than R50 000.

2. We know that Mrs. Sickle now has R13 000 to invest (R14 000 – R1 000).

Therefore:

\[ A = R13\,000 \]
\[ n = 7 \]
\[ i = 33\% \text{ or } 0,33 \]

We can calculate the future value of the investment as follows:

\[ F_S = A \times (1 + ni) \]

\[ = 13\,000 \times (1 + 7 \times 0,33) \]
\[ = 13\,000 \times (1 + 2,31) \]
\[ = 13\,000 \times 3,31 \]
\[ = R43\,030 \]
The amount that Mrs. Sickle will receive after investing R13 000, at a rate of 33% for seven years, is R43 030. Therefore, she will not be able to afford the university fees of R50 000.

Next, we look at a different method of calculating interest, called compound interest.

**Compound interest**

Let's begin by defining the concept of compound interest.

*Compound interest* is interest earned on both the original amount invested *and* the interest earned (accrued) since then. This means that you are also earning interest on the accrued interest.

You have probably heard the phrase 'earning interest on interest'. What does this mean?

In the previous section, you learnt that an investment earning simple interest calculates the value of the interest on the original amount only. So, if you invested R1 000 for three years, at 12% simple interest, then each year you'd earn R120 interest. The value of the interest never changed, because it was always calculated using the original amount invested.

Now, let's look at what happens with simple interest over the three years more carefully. After one year, you'd have R1 000, plus the interest earned for the year, worth R120. Therefore, after the first year, you would actually have R1 120 altogether. After the second year, you would have earned another R120, and so you would have R1 240 altogether. Remember, we only calculate the interest on the original amount that you invested. Finally, after three years, you would have earned another R120, and so would have R1 360.

Now let's suppose that, rather than have one single investment over three years, you decide to manage it as three separate investments, each over one year. After the first year, you would once again have R1 000, plus the R120 worth of interest, totalling R1 120. Now, you can invest this total amount for the second year. So after the second year, you'd have R1 120, plus 12% of R1 120, or R134,40. Therefore, after two years, you'd have R1 254,40. Similarly, you can now invest this total for the third year, this time earning 12% of R1 254,40, which gives you interest of R150,53, and a total of R1 404,93.

Do you see that not only are you earning interest on the original amount of R1 000, but also on the interest earned during the three years? This is what we mean by earning interest on interest, and this is the essence of calculating interest using the compound interest method. *Table 11* shows a comparison of the two methods of calculating interest.

<table>
<thead>
<tr>
<th>End of year</th>
<th>Simple interest (12%)</th>
<th>Compound interest (12%)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Base amount</td>
<td>Interest</td>
</tr>
<tr>
<td>1</td>
<td>R1 000</td>
<td>R120</td>
</tr>
<tr>
<td>2</td>
<td>R1 000</td>
<td>R120</td>
</tr>
<tr>
<td>3</td>
<td>R1 000</td>
<td>R120</td>
</tr>
</tbody>
</table>
Let’s look more closely at the sequence that results from the compound interest calculation. If we let the initial investment amount be \( A \), then the total value of the investment increases as follows:

\[
\begin{align*}
    t_1 &= A \\
    t_2 &= A + t_1 \times 0.12 \\
    &= A + (A \times 0.12) \\
    &= A \times (1 + 0.12) \\
    &= A \times 1.12 \\
    t_3 &= t_2 + t_2 \times 0.12 \\
    &= t_2 \times (1 + 0.12) \\
    &= t_2 \times 1.12 \\
    &= A \times 1.12 \times 1.12 \\
    &= A \times (1.12)^2 \\
    t_4 &= t_3 + (t_3 \times 0.12) \\
    &= t_3 \times (1 + 0.12) \\
    &= t_3 \times 1.12 \\
    &= A \times (1.12)^2 \times 1.12 \\
    &= A \times (1.12)^3
\end{align*}
\]

If we let the interest rate be \( i \), then the general expression for the common ratio is:

\[
(1 + i)
\]

Can you see that the calculation forms a geometric sequence? The first term in the sequence is \( A \), the amount invested, and the common ratio is:

\( (1 + 0.12) = 1.12 \).

The second term in the GS shows the value of the sequence at the end of the first year, and the third term is at the end of the second year. In general, \( t_n \) is equal to the value of the sequence at the end of year \((n - 1)\). Another way of looking at this is to say that a compound interest calculation over \( n \) years, contains \((n + 1)\) terms in the sequence.

Now, remember that the general term for a geometric sequence is:

\[
t_n = ar^{n-1}
\]
If we consider $t_n$ to be the value of the investment after $n$ years, then we know that the sequence has $(n + 1)$ terms, rather than the $n$ terms that you learnt about in Lesson 5. Therefore, in the case of a compound interest calculation, we must multiply the common ratio by the initial amount $n$ times, rather than $(n - 1)$ times.

The formula for calculating a future value based on compound interest is:

$$F_C = A \times (1 + i)^n$$

in which $A$ is the original amount invested, $i$ is the interest rate, and $n$ is the time period.

Note that the $F$ in the formula has a subscript of $C$, indicating that this is the formula for compound interest.

In the formula, $n$ stands for every time that we calculate and add interest to the amount invested. So, for example, if an amount is invested for three years, compounded every three months, then $n$ is equal to 12. The reason is that we are compounding the interest on the amount invested four times every year, or once every three months. Therefore, we compound the amount 12 times over the three years. However, if the question states that the amount invested is compounded annually, then $n$ is equal to the number of years of the investment.

Be careful when working out the value of $n$. The variable $n$ essentially represents the number of times that you must compound the interest on the investment. This won’t necessarily be the same as the number of years in the question.

Let’s test the formula using the figures in Table 11.

**EXAMPLE**

What is the value of an investment of R10 000 after three years, using an interest rate of 15%, compounded annually?

**Answer**

We know that:

- $A = R10 000$
- $n = 3$
- $i = 15\%$ or 0,15

Therefore:

$$F_C = A \times (1 + i)^n$$

$$= 10 000 \times (1,15)^3$$

$$= R15 208,75$$

This method of calculating interest is the most commonly used by banks and other investment institutions.

Now let’s work through another example using compound interest.
EXAMPLE

Suppose your parents deposited an amount of R1 000 in a bank account when you were born. Nobody has withdrawn any money from the account, which has been earning 14% interest compounded annually.

1. If your parents want to give this money to you on your 18th birthday, how much will you get?

Answer

1. First, let's identify the values of the variables in the formula:

   \[ A = R1 \ 000 \]
   \[ n = 18 \]
   \[ i = 14\% \text{ or } 0.14 \]

   We can calculate the future value of the investment as follows:

   \[ FC = A \times (1 + i)^n \]
   \[ = 1 \ 000 \times (1 + 0.14)^{18} \]
   \[ = 1 \ 000 \times (1.14)^{18} \]
   \[ = 1 \ 000 \times 10,5751 \]
   \[ = R10 \ 575.16918 \]

   You will receive R10 575.17 when you turn 18.

In the next section, we look at nominal and effective interest rates.

Activity 14

A car that cost R90 000 depreciates at 18% per annum.

1. Determine the book value of the car after five years if depreciation is calculated according to the straight-line method.

2. Find the rate, according to the reducing-balance method, which will yield the same book value as in Question 1, after five years.

Answer

1. \[ FS = A \times (1 - n \times \frac{i}{100}) \]
   \[ = 90 \ 000 \times (1 - 0.18 \times 5) \]
   \[ = 90 \ 000 \times (1 - 0.9) \]
   \[ = 90 \ 000 \times 0.1 \]
   \[ = R9 \ 000.00 \]
Nominal and effective interest rates

Before we continue, let's define both nominal and effective interest rates.

Nominal interest rates

A nominal interest rate is the contractual annual rate of interest charged by a lender or promised by a broker, that is an interest rate that we must compound at a time interval other than a year (generally less than a year). However, we express the rate as an annual rate without taking the full effect of compounding into account.

For example, a nominal annual rate of 8%, based on quarterly compounding, is actually a rate of 2% \(\left(\frac{8}{4}\right)\), compounded quarterly. Similarly, a nominal annual rate of 20%, based on half-yearly compounding, is actually a rate of 10% \(\left(\frac{20}{2}\right)\), compounded half-yearly. Importantly, the nominal rates are the same, but if you were to invest the same amount of money in each option, then the option based on quarterly compounding would give a better return, as we'll see later.

Let's look at nominal rates from the other direction. Suppose you were told that an investment yielded a return of 5%, compounded every four months. What would the nominal annual rate be? To calculate the nominal rate, all you need to do is to multiply the percentage by the number of months necessary to adjust the period to be equal to a year. So in this case, the nominal annual rate would be 15% \((5\% \times 3)\).

As we mentioned a little earlier, comparing two interest rates that have the same nominal annual rate, but that have different compounding periods, is not straightforward. We need a way of equalising the two, so that we can more easily determine which is actually the better option. The way to do this is to use an effective interest rate.

Effective interest rates

An effective interest rate is the annual rate of interest actually paid or earned, that is it takes into account the full effects of compounding.

By including the full effects of compounding in the effective rate, we now have a way of directly comparing two interest rates.
Remember that at the heart of the concept of compound interest is the idea of earning interest on interest. Consider our earlier example of the interest rate of 7%, compounded half-yearly. If we invest R100, then after six months we’ll have R107. We’ve earned interest of 7% of R100, or R7. Then, at the end of the year, we’ll earn 7% interest of R107, not R100. In other words, in the second half of the year, we earn interest on the interest that we earned in the first half of the year. In this case, the value of the interest in the second half of the year is R107 $\times$ 7%, which is R7.49. So, by the end of the year, we’ve actually earned a total of R14.49 in interest.

If we simply multiply the 7% by two so that it becomes a nominal annual rate, then we arrive at a rate of 14% for the year. Our investment of R100 would then yield R14 interest at the end of the year. As you can see, simply multiplying by two doesn’t take into account the compounding that occurred in the middle of the year. However, if we calculated the effective interest equivalent to 7% compounded half-yearly, then we’d have a far more accurate value. Let’s see how to do this.

**Formula for the effective rate (ER)**

The formula to calculate an effective interest rate is as follows:

$$ER = \left(1 + \frac{NR}{n}\right)^n - 1$$

where $ER$ is the effective rate, $NR$ is the nominal rate, and $n$ is the number of quoted time periods.

The next example shows how to calculate the effective rate.

**EXAMPLE**

What would the effective rate be if the nominal rate is 16% and interest is compounded monthly for one year?

**Answer**

There are 12 Months in a year, so $n = 12$.

Therefore:

$$NR = 16\% \text{ or } 0.16$$

$$ER = \left(1 + \frac{NR}{n}\right)^n - 1$$

$$= \left(1 + \frac{0.16}{12}\right)^{12} - 1$$

$$= (1.0133)^{12} - 1$$

$$= 1.17227 - 1$$

$$= 0.17227$$

Therefore, the effective rate is 17.2%. This is 1.2% more than the nominal rate of 16%.

Now work carefully through the following activity to practise what you have learned about effective interest rates.
Activity 15

Calculate which of the two investments would yield a better financial return.

- an investment of 8.2% compounded semi-annually; or
- an investment of 8.15% compounded quarterly

Answer

To determine which investment has the better interest rate, we have to compare their effective interest rates.

For the first option of 8.2% compounded semi-annually, you have the following values for the formula:

\[ n = 2 \text{ (there are two intervals in each year)} \]
\[ NR = 8.2\%, \text{ or } 0.082 \]

Therefore:

\[
ER = \left(1 + \frac{NR}{n}\right)^n - 1
\]
\[
= \left(1 + \frac{0.082}{2}\right)^2 - 1
\]
\[
= (1.041)^2 - 1
\]
\[
= 0.083681
\]

Therefore, the effective interest rate for the first option is 8.37%.

For the second option of 8% compounded quarterly, you have the following values for the formula:

\[ n = 4 \text{ (there are four quarters in one year)} \]
\[ NR = 8.15\%, \text{ or } 0.0815 \]

Therefore:

\[
ER = \left(1 + \frac{NR}{n}\right)^n - 1
\]
\[
= \left(1 + \frac{0.0815}{4}\right)^4 - 1
\]
\[
= (1.020375)^4 - 1
\]
\[
= 0.084024\ldots
\]

Therefore, the effective interest rate for the first option is 8.40%.

Since the second option has a higher effective rate than the first option, we can safely say that the second option will yield a better return.
Naturally, using effective rates to compare interest rates isn’t limited to just comparing two different rates. Now try the following activity, in which you need to make a choice between three different interest rates.

**Activity 16**

Calculate and compare the effective rates for the following three nominal rates:

1. Bank A: 12% compounded bi-annually (every six months)
2. Bank B: 11% compounded quarterly
3. Bank C: 11.85% compounded monthly

**Answer**

1. \( ER \) for Bank A
   \[
   \left(1 + \frac{0.12}{2}\right)^2 - 1
   \]
   \[
   = (1.06)^2 - 1
   \]
   \[
   = 0.124 \text{ or } 12.4\%
   \]
2. \( ER \) for Bank B
   \[
   \left(1 + \frac{0.11}{4}\right)^4 - 1
   \]
   \[
   = (1.028)^4 - 1
   \]
   \[
   = 0.117 \text{ or } 11.7\%
   \]
3. \( ER \) for Bank C
   \[
   \left(1 + \frac{0.119}{12}\right)^{12} - 1
   \]
   \[
   = (1.01)^{12} - 1
   \]
   \[
   = 0.127 \text{ or } 12.7\%
   \]

From the point of view of the investor, Bank C offers the best interest rate. However, from the borrower’s point of view, Bank B charges the best interest rate.

Now let’s explain the conclusion in Activity 16 above.

When you invest money, you want the highest possible interest rate in order to get the most interest from your invested money. Therefore, Bank A offers the best option. Just remember that high interest rates can sometimes mean a risky investment!

But, when you borrow money, you want the lowest possible interest rate to ensure that your debt grows as slowly as possible. That means you have to pay back less money overall. In this case, Bank B offers the best alternative.

Interest rates expressed as nominal interest rates are not easy to compare. The only time you can compare two nominal rates is if the compounding periods are the same. In all other cases, you first need to convert the rates to an effective rate before drawing any conclusions.
In last part of this section, we'll look at how to determine a nominal rate, given a particular effective rate.

**Formula for the nominal rate (NR)**

Recall that the formula for the effective interest rate \((ER)\) is:

\[
ER = \left(1 + \frac{NR}{n}\right)^n - 1
\]

By solving for \(NR\), we can change this formula to:

\[
NR = n \times \left(\sqrt[n]{ER} + 1 - 1\right)
\]

in which \(ER\) is the effective rate, \(NR\) is the nominal rate, and \(n\) is the number of quoted time periods.

You may also see this formula written in a slightly different way, as follows:

\[
NR = n \times \left((ER + 1)^\frac{1}{n} - 1\right)
\]

There are two ways of writing an \(n\)th root of a value, namely \(\sqrt[n]{x}\) and \(x^\frac{1}{n}\). The two methods mean exactly the same thing. In this study unit, we'll use the first of the two notations, as this is the format commonly found on calculators.

Let's see how we can put this formula into practice. Notice how interest rates can be deceiving. You are quoted a particular rate, but because it's compounded quarterly, you effectively earn (or pay!) more interest.

The next example takes the view of someone lending money, rather than investing money. In this case, you'll see how the lender is able to quote a seemingly low interest rate on a loan that he offers, when in fact the effective interest rate is noticeably higher. It's for this reason that you always need to be careful when evaluating an interest rate.

**EXAMPLE**

Thobeka owns a business that provides small loans. He wants to earn an effective interest rate of 30% per annum on a particular loan, but wants to quote a nominal rate that reflects quarterly compounding. Which rate does he quote?

**Answer**

We need to calculate \(NR\), with \(n = 4\) and \(ER = 30\% \text{ or } 0,3\).

Now:

\[
NR = n \times \left(\sqrt[n]{ER} + 1 - 1\right)
\]

\[
= 4 \times \left(\sqrt[4]{0,3} + 1 - 1\right)
\]

\[
= 0,271159 \text{ or } 27,16\%
\]

Thobeka needs to quote an interest rate of 27,16% compounded quarterly.

In the above example, the nominal rate of 27,16% and the effective rate of 30% per annum are equivalent. However, by using the nominal rate, Thobeka is able to quote a rate that *sounds* lower, even though it's not.
Remember, the only way to determine the true value of an interest rate is to calculate the effective rate.

Now let’s see how the formula for compound increase applies to appreciation.

**Appreciation**

What is appreciation?

Appreciation is an increase in value over time.

Land and buildings are examples of assets that appreciate in value over time.

You can use the same formula that we used for compound interest to calculate the value of an asset that appreciates in value. Remember that the formula for compound interest is:

$$F_V = P_V \times (1 + i)^n$$

In this case, $i$ represents the rate of appreciation, rather than the interest rate. The variable $A$ represents the initial value of the asset, and $n$ is the period of time in question.

Let’s work through an example of appreciation.

**EXAMPLE**

Mrs. Sickle bought a house for R500 000 in 2005. If the appreciation rate for the house is 8% per annum, how much is her house worth in 2009?

**Answer**

We need to use the formula for appreciation to calculate the value of Mrs. Sickle’s house after four years.

We know that:

$$P_V = 500 000$$

$$i = 0,08 \ (8\%)$$

$$n = 4 \ (From \ 2005 \ to \ 2009 \ is \ 4 \ years)$$

Therefore:

$$F_V = P_V \times (1 + i)^n$$

$$F_C = 500 000 \times (1 \ 0,08)^4$$

$$= R \ 680 \ 244,48$$

Therefore, Mrs Sickle’s house will be worth R680 244,48 after four years.

In the next section, we’ll look at population growth as a final example of compound increase.
Population growth

Population growth is yet another example of compound increase. The term population refers to the number of people, animals or other life forms that inhabit a particular space.

Once again, we can use the same formula as before.

\[ F_V = P_V \times (1 + i)^n \]

In this case, \( i \) represents the rate of population growth, rather than the interest rate. The variable \( A \) represents the initial size of the population, and \( n \) is the period of time in which the population experiences growth.

EXAMPLE

The population of a certain country grows at a rate of 2% per year.

1. If the population was 18,6 million at the beginning of 1990, what will it be at the end of 2010?
2. How long will it take the population to triple?

Answer

1. We need to use the formula for population growth to calculate the size of the population at the end of 2010.

   We know that:
   
   \[
   A = 18,6 \text{ million} \\
   i = 0,02 (2\%) \\
   n = 21 \text{ (Remember, the end of 2010 is equivalent to start of 2011)}
   \]

   Therefore:
   
   \[
   F_V = P_V \times (1 + i)^n \\
   = 18,6 \times (1 + 0,02)^{21} \\
   = 23,5892
   \]

   Therefore, the population will be about 23,6 million at the end of 2010.

2. We need to use the formula for population growth, and to solve for \( n \). We know that:

   \[
   A = 18,6 \text{ million} \\
   i = 0,02 (2\%) \\
   F_C = 55,8 \text{ million} (18,6 \times 3 = 55,8)
   \]
Therefore:

\[ F_C = A \times (1 + i)^n \]

\[ 55,8 = 18,6 \times (1 + 0,02)^n \]

\[ 3 = (1,02)^n \]

\[ \log 3 = n \log (1,02) \]

\[ n = \frac{\log 3}{\log (1,02)} \]

\[ = 55,48 \]

Therefore, the population will triple in 55,48 years.

Now let’s move on to the concept of **compound decrease**.

**Compound decrease**

There are three examples of compound decrease that we’ll discuss in the section, namely:

- depreciation;
- radioactive decay; and
- inflation.

The formula for compound decrease is a slight variation on that for a compound increase. The formula is:

\[ F_C = A \times (1 - i)^n \]

where \(A\) is the original amount invested, \(i\) is the rate of decrease, and \(n\) is the time period. (This is also called the reducing-balance method.)

The only difference between this formula and the one for a compound increase is the minus sign that replaces the plus sign inside the brackets. This should make sense, because in the case of a compound decrease, the original value is getting smaller, not bigger.

**Depreciation**

*Depreciation* is a decrease in value over time.

We usually express the depreciation of an asset as a percentage per year. Let’s look at an example of how to calculate the value of an asset that depreciates in value.
EXAMPLE

The value of a vehicle depreciates by 20% per year. If the current value of the vehicle is R100 000, then:

1. Calculate its value after five years.

Answer

1. You know the following information:

   \[ A = 100\,000 \]
   \[ i = 0,2 \text{ (20\%)} \]
   \[ n = 5 \]

   Therefore:

   \[ FV = PV \times (1 - i)^n \]
   \[ FC = A \times (1 - i)^n \]
   \[ = 100\,000 \times (1 - 0,2)^5 \]
   \[ = 32\,768 \]

   Therefore, the value of the vehicle after five years is R32 768.

Another good example of a compound decrease is the decay of a radioactive element.

Radioactive decay

Radioactive decay occurs when an unstable atomic nucleus loses energy by emitting radiation. The rate at which the atom loses energy is an exponential function.

EXAMPLE

The radioactive isotope sesium-137 decays at a rate of 1,86% per annum (this means that 1,86% of the sesium-137 present at the beginning of the year will decay during the year).

1. If you start today with 100 g sesium-137, calculate the mass remaining after ten years. Give your answer to two decimal places.

2. Calculate how many years it will take before 50 g sesium-137 is left. We call this period the half-life of sesium-137.

Answer

1. Let the mass remaining after \( k \) years be \( M_k \). You know the following information:

   \[ A = 100 \]
   \[ i = 0,0186 \text{ (1,86\%)} \]
   \[ n = 10 \]
Therefore:

\[ M_{10} = A \times (1 - i)^n \]
\[ = 100 \times (1 - 0,0186)^{10} \]
\[ = 82,88 \]

Therefore, the mass remaining after five years is 82,88 g.

2. To calculate the half-life of cesium-137 we must solve the following exponential equation:

\[ 50 = 100 \times (1 - 0,0186)^k \]
\[ 50 = 100 \times (0,9814)^k \]
\[ 0,5 = (0,9814)^k \]
\[ \log 0,5 = k \log 0,9814 \]
\[ k = \frac{\log 0,5}{\log 0,9814} \]
\[ = 36,92 \]

Therefore, it will take 36,92 years before only 50 g of cesium-137 remains.

Lastly, let’s look at inflation as an example of a compound decrease calculation.

**Inflation**

Inflation is a term used in economics.

The Oxford English Dictionary defines *inflation* as ‘a general increase in prices and fall in the purchasing value of money’.

You may be wondering why we classify inflation as a compound decrease if prices are increasing. The problem with inflation in an economy is that increased prices mean that you can buy less for the same amount of money. For example, R100 10 years ago could buy much more than R100 can today. So this is what we mean by a fall in the purchasing value of money. The same amount of money can buy less than it could before.

**EXAMPLE**

If the inflation rate remains constant over the next five years at 10% per annum, what will the value of an amount with current value R10 000 be after five years?

**Answer**

We know the following information:

\[ A = 10 \ 000 \]
\[ i = 0,1 \ (10\%) \]
\[ n = 5 \]
Therefore:

\[ FC = A \times (1 - i)^n \]

\[ = 10\,000 \times (1 - 0,1)^5 \]

\[ = 5\,904,90 \]

Therefore, R10 000 will be worth R 5 904,90 in five years time.

As you can see, inflation decreases the value of your money. Try to think of something that you can buy now for about R 5 904,90. According to the calculation in the previous example, that same product will cost you R10 000 in five years time. That is the effect of inflation.

Now work carefully through the following activity to practise what you have learnt about compound increase and decrease.

### Activity 17

1. An amount of R2 000 is invested at 15% per annum, compounded monthly. Calculate the value of the investment after five years.

2. A person wants to make an investment at 10% per annum, which must have a value of R25 000 after eight years. Calculate the amount to be invested if the interest is compounded:
   
   (a) annually
   
   (b) monthly.

3. A vehicle depreciates within five years from R100 000 to R25 000. Calculate the depreciation rate (to one decimal place).

**Answer**

1. Interest is compounded monthly, or 12 times each year. That means that, over 5 years, interest is compounded 12 \times 5 = 60 times.

   The interest rate is given as an annual rate, but, because the compounding occurs each month, you need to calculate the monthly rate. You do so by dividing the annual rate by 12.

   Therefore:

   \[ A = R\,2\,000 \]

   \[ n = 60 \]

   \[ i = \frac{15}{12} \%

   \[ = 1,25\% \text{ or } 0,0125 \]
You can calculate the future value of the investment as follows:

\[ F_C = A \times (1 + i)^n \]

\[ = 2000 \times (1 + 0,0125)^{60} \]

\[ = 2000 \times (1,0125)^{60} \]

\[ = 4214,36 \]

Therefore, the investment will be worth R4 214,36 in five years' time.

2. (a) First, identify the values of the variables in the formula:

\[ A = ? \]
\[ n = 8 \]
\[ i = 10\% \text{ or } 0,1 \]
\[ F_C = 25000 \]

In this case, you need to solve for \( A \) in the following equation:

\[ F_C = A \times (1 + i)^n \]

\[ 25000 = A \times (1,1)^8 \]

\[ A = \frac{25000}{(1,1)^8} \]

\[ = 11662,68 \]

Therefore, the person would need to invest R11 662,68.

(b) First, identify the values of the variables in the formula:

\[ A = ? \]
\[ n = 8 \times 12 \]
\[ = 96 \]
\[ i = \frac{10}{12} \% \]

\[ = 0,83\% \text{ or } 0,008 \, 3 \]
\[ F_C = 25000 \]

In this case, you need to solve for \( A \) in the following equation:

\[ F_C = A \times (1 + i)^n \]

\[ 25000 = A \times (1,008 \, 3)^{96} \]

\[ A = \frac{25000}{(1,008 \, 3)^{96}} \]

\[ = 11306,35 \]

Therefore, the person would need to invest R11 306,35.
3. You know the following information:

\[ A = 100\,000 \]

\[ i = ? \]

\[ n = 5 \]

\[ F_C = 25\,000 \]

Therefore:

\[ F_C = A \times (1 - i)^n \]

\[ 25\,000 = 100\,000 \times (1 - i)^5 \]

\[ \frac{1}{4} = (1 - i)^5 \]

\[ \sqrt[5]{\frac{1}{4}} = 1 - i \]

\[ i = 1 - \sqrt[5]{\frac{1}{4}} \]

\[ = 0.24214 \ldots \]

\[ = 24.2\% \]

The depreciation rate is 24.2%.

---

**Summary**

In Lesson 8, you learnt about compound increase and decrease.

Compound increase and decrease functions are exponential in nature, and so represent geometric sequences of values over time.

The first part of the lesson focused on interest calculations. The type of interest that you’ll encounter in everyday life is called compound interest. However, in order to fully understand compound interest, you first need to know what simple interest is. We also examined the difference between nominal and effective rates of interest. Here, you learnt how banks and other financial institutions are able to offer interest rates that sound very good, but in fact are not all that good at all. The only way to make a direct comparison between two or more interest rates is to convert all of them to effective interest rates.

The examples of compound increase that we discussed were:

- interest rates;
- appreciation; and
- population growth.
Then, the examples of compound decrease that we discussed were:

- depreciation;
- radioactive decay; and
- inflation.

## Self-assessment Questions 8

Test your knowledge of this lesson by completing the self-assessment questions below. When you answer the questions, don’t look at the suggested answers that we give straight away. Look at them only after you’ve written your answers down, and then compare your answers with our answers.

1. On a boy’s eighteenth birthday, R50 000 is invested for him at 18% per year, compounded monthly. What will his age be (to the nearest year) when the value of the investment is one million rand?

2. On a girl’s tenth birthday, R5 000 is invested at 11% per year, compounded every six months. What will the value of the investment be on her eighteenth birthday?

3. An investment of R10 000 grows threefold in six years. Calculate the interest rate if interest is compounded monthly.

4. At the end of six years, the value of an amount of money invested at 10% per year, compounded annually, is R115 891,60. Calculate the value of the initial investment (to the nearest rand).

5. A vehicle depreciates at 15% per year. How long (to the nearest year) will it take for the value of the vehicle to halve?

## Suggested answers to Self-assessment Questions 8

1. First, identify the values of the variables in the formula.

   Interest is compounded monthly, or 12 times each year. Therefore, over five years, interest is compounded $12 \times 5 = 60$ times.

   The interest rate is given as an annual rate, but, because the compounding occurs each month, you need to calculate the monthly rate. You do so by dividing the annual rate by 12.

   Therefore:

   $$ A = 50\,000 $$

   $$ n = ? $$

   $$ i = \frac{18}{12} \% $$

   $$ = 1,5\% \text{ or } 0,015 $$

   $$ FC = 1\,000\,000 $$
You need to solve for \( n \) in the following equation:

\[
F_C = A \times (1 + i)^n
\]

\[
1 000 000 = 50 000 \times (1,015)^n
\]

\[
20 = (1,015)^n
\]

\[
n = \frac{\log 20}{\log (1,015)}
\]

\[
= 201,21
\]

Therefore, there are 201,21 compound intervals. However, interest is compounded monthly. So these intervals represent \( \frac{201,21}{12} \), or 16,77 years.

Therefore, if round this value off to 17 years (the nearest year), then the boy will be 35 when the value of the investment is R1 000 000.

2. The length of time of the investment is four years. Interest is compounded twice a year. Therefore, over four years, interest is compounded eight times.

The interest rate is given as an annual rate, but, because the compounding occurs every six months, you need to calculate the monthly rate. You do so by dividing the annual rate by 2. Therefore:

\[
A = 5 000
\]

\[
n = 16
\]

\[
i = \frac{11}{2}\% = 5,5\% \text{ or } 0,055
\]

Therefore:

\[
F_C = A \times (1 + i)^n
\]

\[
= 5 000 \times (1,055)^{16}
\]

\[
= 11 776,31
\]

Therefore, the investment will be worth R11 776,31 in four years time.

3. Identify the values of the variables in the formula.

The length of time of the investment is six years. Interest is compounded monthly. Therefore, there are \( 6 \times 12 = 72 \) time intervals.

\[
A = 10 000
\]

\[
n = 72
\]

\[
i = ?
\]

\[
F_C = 30 000
\]
Therefore:

\[ F_C = A \times (1 + i)^n \]

\[ 30000 = 10000 \times (1 + i)^{72} \]

\[ 3 = (1 + i)^{72} \]

\[ \sqrt[72]{3} = 1 + i \]

\[ i = \sqrt[72]{3} - 1 \]

\[ i = 0.0153755\ldots \]

\[ i = 1.5\% \]

Therefore, since this is a monthly interest rate, the annual interest rate is 1.5%.

4. Identify the values of the variables in the formula.

\[ A = ? \]

\[ n = 6 \]

\[ i = 10\% \text{ or } 0.1 \]

\[ F_C = 115891.60 \]

Therefore:

\[ F_C = A \times (1 + i)^n \]

\[ 115891.60 = A \times (1.1)^6 \]

\[ A = \frac{115891.60}{(1.1)^6} \]

\[ = 65418 \]

Therefore, the original investment was R65 418.

5. Identify the values of the variables in the formula. Let the initial value of the vehicle be equal to \( a \).

\[ A = a \]

\[ n = ? \]

\[ i = 15\% \text{ or } 0.15 \]

\[ F_C = \frac{1}{2}a \]
You need to solve for \( n \) in the following equation:

\[
F_C = A \times (1 - i)^n
\]

\[
\frac{1}{2}a = a \times (1 - 0.15)^n
\]

\[
\frac{1}{2} = (0.85)^n
\]

\[
n = \frac{\log 0.5}{\log (0.85)}
\]

\[
= 4.27
\]

Therefore, it will take 4.27 years for the value of the vehicle to halve. If you round this off to the nearest year, then it will take approximately four years for the value to halve.

**Check your competence**

Now that you have worked through this lesson, please check that you can perform the tasks below:

- I can use simple and compound decay formulae (\( A = P(1 - ni) \) and \( A = P(1 - i)^n \)) to solve problems (including, straight line depreciation and depreciation on a reducing balance).
- I can demonstrate an understanding of different periods of compounding growth and decay (including, effective compounding growth and decay and including effective and nominal interest rates).
- I can calculate the value of \( n \) in the formula \( A = P(1 \pm i)^n \).
- I can solve non-routine, unseen problems.

**The next lesson**

If you are sure that you understand the contents covered in this lesson, and have achieved your Assessment Standards, start Lesson 9.
Learning Outcomes for Lesson 9

After you have worked through Lesson 9, you should be able to do the following:

<table>
<thead>
<tr>
<th>LO</th>
<th>Assessment Standards</th>
</tr>
</thead>
</table>
| LO 1 | AS 12.1.4 (a) Calculate the value of n in the formula  
|      | \( A = P (1 \pm i)^n \). |
|      | (b) Apply knowledge of geometric series to solving  
|      | annuity, bond repayment and sinking fund  
|      | problems, with or without the use of the formulae:  
|      | \( F = \frac{x(1 + i)^n - 1}{i} \) and  
|      | \( P = \frac{x(1 - (1 + i)^{-n})}{i} \) |
|      | AS 12.1.5 Critically analyse investment and loan options, and make  
|      | informed decisions as to the best option(s) (including,  
|      | pyramid and micro-lenders’ schemes). |
|      | AS 12.1.6 Solve non-routine, unseen problems. |

We would like to help you to remain in complete control of your studies. So we give you an opportunity to check your competence at the end of this lesson.

Introduction

In Lesson 8, you learnt about compound increase and decrease and, in particular, you learnt how these concepts apply to interest rates. Compound interest is something that affects all our lives every day. In this lesson, you’ll learn more about how compound interest applies to various types of financial calculations.

Except for a few wealthy individuals, most of us do not have the financial means to buy everything we need with cash. Most of us rely on credit facilities offered by retailers and banks to finance our purchases. This is applicable particularly to furniture, cars and houses. In this lesson, we will discuss the following credit agreements:

- hire-purchase agreements; and
- mortgage loans.

When we make provision for our retirement, we usually make regular payments towards a retirement annuity in order for us to meet our financial needs at that future date. Another option is to create a savings account, and to make regular deposits into that account. Therefore, in this lesson, we’ll also examine retirement annuities and savings in some detail.
The topics covered in this lesson are:

- hire-purchase agreements;
- mortgage loans;
- regular savings; and
- retirement annuities.

Let’s begin by looking at hire-purchase agreements.

**Hire-purchase agreements**

When we buy assets on credit, we may enter into hire-purchase agreements with our creditors. What is the nature of a hire purchase contract?

A *hire-purchase contract* is a financing arrangement that enables a person to take possession of an asset, while making regular payments on the asset. The asset only becomes the legal property of the person after making the final payment on the asset.

Hire purchase is an expensive option in the long term. Normally a deposit is required when buying an asset on hire purchase. Although the instalment of hire-purchase agreements is usually low, the total amount that you eventually pay back is often much higher than the cash price of the asset.

We will illustrate this fact in the following example.

**EXAMPLE**

You want to buy furniture by means of a hire-purchase agreement. The cash price of the furniture amounts to R10 000. The furniture store requires a 10% deposit on the cash price, and equal payments of R800 per month over 24 months.

1. How much do you actually pay for the furniture?
2. How much interest do you pay in total over the 24 months?

**Answer**

1. We have that:

   \[
   \text{total paid} = \text{deposit} + (\text{instalment} \times \text{number of periods})
   \]

   \[
   = 1 000 + (800 \times 24)
   \]

   \[
   = 20 200
   \]

   Therefore, the total amount paid for the furniture is R20 200.

2. The total amount of interest paid is the difference between the total amount paid and the original cash price. Therefore:

   \[
   \text{total interest paid} = \text{total paid} - \text{cash price}
   \]

   \[
   = R20 200 - R10 000
   \]

   \[
   = R10 200
   \]

   As you can see, the amount of interest you will pay over 24 months is more than the price of the furniture itself!
To avoid paying such excessive amounts of money unnecessarily, it’s usually best to avoid buying on credit. It is usually much cheaper for you to buy assets for cash, but not all of us have the money readily at hand. We need to save money in order to buy the things that we need. By saving, we earn interest on our money and watch it increase over time!

We will examine savings more closely later on in this unit. First, let’s see how to calculate an instalment on a hire-purchase agreement.

**Calculating instalment payments when buying on credit**

The formula for calculating an instalment \((S)\) when buying on credit is:

\[
S = \frac{A \times i \times (1 + i)^n}{(1 + i)^n - 1}
\]

in which:

- \(A\) is the initial amount to be borrowed;
- \(i\) is the interest rate charged per period; and
- \(n\) is the number of time periods over which instalments are paid.

Although this formula may look complicated, it becomes more manageable when used in practice a few times.

Notice that the interest rate \((i)\) can be given as a yearly, bi-annually, quarterly or monthly rate. This has implications on the number of time periods \((n)\) in the formula. Make sure that \(i\) and \(n\) are always consistent. For instance, if your interest rate is a monthly rate, then \(n\) must reflect the number of months.

Look at the following example as an illustration of how to apply the formula for calculating an instalment.

---

**EXAMPLE**

Suppose you want to buy a car for R65 000. The motor dealer offers you financing, and tells you he will give you the special rate of 1% per month, over 54 months. Although you know this is no special rate, you need to know the costs involved.

1. What is the monthly instalment?
2. What is the total amount that you have paid over 54 months?
3. How much interest will you have paid by the end of the 54-month period?

**Answer**

1. Firstly, indicate the values that the formula needs:
   
   \[
   A = R65 000 \\
   i = 1\% \text{ per month} \\
   n = 54 \text{ months}
   \]
Now you can use the formula to calculate the monthly instalment:

\[ S = \frac{A \times i \times (1 + i)^n}{(1 + i)^n - 1} \]

\[ = \frac{65 \, 000 \times 0,01 \times (1,01)^{54}}{(1,01)^{54} - 1} \]

\[ = \frac{65 \, 000 \times 0,01 \times 1,711}{1,711 - 1} \]

\[ = \frac{1112,15}{0,711} \]

\[ = 1 \, 564,21 \]

The monthly instalment for the new car will be R1 564,21.

2. The total amount you will have paid after 54 months is:

\[ \text{total paid} = S \times n \]

\[ = 1 \, 564,21 \times 54 \]

\[ = 84 \, 438,72 \]

Therefore, you will have paid a total of R84 438,72 after 54 months.

3. The interest that you will have paid by the end of the 54-month period is the difference between the total amount paid and the original cash price of the car.

\[ \text{total interest paid} = \text{total amount paid} - \text{cash price} \]

\[ = 84 \, 438,72 - 65 \, 000 \]

\[ = 19 \, 438,72 \]

As you can see, when you are able to do these calculations, you are able to decide for yourself if you would like to buy the car, and what the real costs are.

Now work carefully through the following activity to practise what you have learnt about hire-purchase contracts.
Activity 18

Go to a furniture store. Find one item that you like, and find out the hire-purchase terms of payment. Compare this to the cash price, and calculate how much interest you would pay on the hire-purchase option.

Answer

Of course, each person will have a different answer to this question. However, we’re giving you a model answer against which to compare your answer. Suppose the store sells a television at a cash price of R2 999. The hire-purchase agreement for the television is R300 per month, for 12 months.

\[
\text{total paid} = 300 \times 12
\]

\[
= 3600
\]

\[
\text{interest paid} = \text{total paid} - \text{cash price}
\]

\[
= 3600 - 2999
\]

\[
= 601
\]

Therefore, you will pay a total amount of R3 600 for the TV, which is only worth R2 999.

Hire-purchase agreements generally apply to amounts that you will pay off over a few years. However, if you buy a property by means of a mortgage loan, then the payments are over a much longer time. As you’ll see in the next section, the principles are much the same.

Mortgage loans

Mortgage loans are also known as home loans. Since most of us either dream about owning a home, or own a home already, home loans are of interest to us.

A mortgage loan is a loan that a person takes out for the specific purpose of buying fixed property. Examples of fixed property include a piece of land or a dwelling built on it.

Most home loans are repaid over a term of 20 years. This is a long time to commit to regular monthly payments, and the bank charges a huge amount of interest over this period. Therefore, we are very interested in the calculations of interest, monthly repayments and the term (payment period) of a home loan.

It is important to understand that when we sign a home loan agreement with a bank, the bank retains the right to seize the house and sell it if we do not make the required monthly repayments. For this reason too, it is important and useful to understand how the repayments on a mortgage loan are calculated.

The formula for calculating the monthly repayment on a mortgage loan is the same as the one we used to calculate instalments on credit purchases.
Remember, the formula for calculating a hire-purchase instalment \((S)\) is:

\[
S = \frac{A \times i \times (1 + i)^n}{(1 + i)^n - 1}
\]

In the case of mortgage loans:

- \(A\) is the amount that you borrow (usually the price of the house);
- \(i\) is the interest rate charged per period (usually a monthly rate); and
- \(n\) is the number of time periods over which instalments are paid.

Let’s look at an example in which we calculate monthly repayments on a home loan, and demonstrate how we are able to save on interest by paying off our home loan more quickly.

**EXAMPLE**

You apply for a mortgage loan of R400 000 to buy a house. The bank grants you the full loan amount of R400 000 at an interest rate of 1% per month, to be paid off over a term of 20 years.

1. What is your monthly repayment?
2. How much interest will you pay over the 20-year period?
3. What would you have to pay each month to pay off the mortgage loan in 15 years?
4. How much interest will you have paid by the end of the 15-year period?
5. How much would you save in interest if paid the loan off over 15 years instead of 20 years?

**Answers**

The first step is to make sure that all the variables are consistent. The repayments are monthly, so we need to make sure that the interest rate \((i)\) and the number of time periods \((n)\) are in monthly units.

- \(i = 1\%\) per month (already in \%/month)
- \(n = 20 \times 12\)
  \[= 240\text{ months}\]

1. You have the following figures:

   \[
   A = \text{R}400\ 000
   \]
   \[
   i = 1\%\text{ per month}
   \]
   \[
   n = 240\text{ months}
   \]
Therefore, the monthly repayment ($S$) is:

$$S = \frac{A \times i \times (1 + i)^n}{(1 + i)^n - 1}$$

$$= \frac{400 000 \times 0,01 \times (1,01)^{240}}{(1,01)^{240} - 1}$$

$$= \frac{400 000 \times 0,01 \times 10,893}{10,893 - 1}$$

$$= \frac{43 572}{9,893}$$

$$= 4 404,33$$

Therefore, the monthly instalments are R4 404,33.

2. The interest that you will pay is the difference between the total amount you pay and the original price of the home. The total amount you’ll pay over 20 years is:

$$\text{total paid} = S \times n$$

$$= 4 404,33 \times 240$$

$$= 1 056 799,20$$

$$\text{total interest paid} = \text{total paid} - \text{original price}$$

$$= 1 056 799,20 - 400 000$$

$$= 656 799,20$$

Therefore, the interest you will pay at 1% per month after 20 years for buying a house worth R400 000 is R656 799,20, which is more than the value of the house!

3. The calculation is the same as in the first question, except that the period is now 15 years, rather than 20 years. You have the following figures:

$$A = \text{R}400 000$$

$$i = 1\% \text{ per month}$$

$$n = 15 \times 12$$

$$= 180 \text{ months}$$

Therefore, the monthly repayment ($S$) is:

$$S = \frac{A \times i \times (1 + i)^n}{(1 + i)^n - 1}$$

$$= \frac{400 000 \times 0,01 \times (1,01)^{180}}{(1,01)^{180} - 1}$$
Therefore, the monthly instalments are R4 800,64.

4. The interest that you have paid is the difference between the total amount you pay and the original price of the home. The total amount you'll pay over 15 years is:

\[
\text{total paid} = S \times n
\]
\[
= 4800,64 \times 180
\]
\[
= 864\,115,20
\]

\[
\text{total interest paid} = \text{total paid} - \text{original price}
\]
\[
= 864\,115,20 - 400\,000
\]
\[
= 464\,115,20
\]

Therefore, the interest you will pay at 1% per month after 15 years for buying a house worth R400 000 is R464 115,20.

5. The interest you save by paying off the home in 15 years, rather than 20 years is:

\[
\text{interest saved} = \text{interest over 20 years} - \text{interest over 15 years}
\]
\[
= 656\,799,20 - 464\,115,20
\]
\[
= 192\,684
\]

Therefore, you save R192 684 by paying off your home in 15 years instead of 20 years.

This example illustrates that, by paying R4 800 per month rather than R4 404 per month, you can save R192 921 in interest! You will also have finished paying off your house five years earlier. You can then put the R4 800 per month towards another investment or buying something you’ve always wanted!

Now try to do some home loan calculations yourself.
Activity 19

Imagine that you have been granted a mortgage loan of R300 000, with interest charged at 1% per month. Your monthly repayment is R3 300.

What part of the first payment that you make (of R3 300) is used to pay the bank interest, and by what amount is the R300 000 home loan reduced?

Answer

Firstly, calculate the amount of interest that you have to pay at the end of the first month.

This amount is:

\[
\text{interest} = 300 \,000 \times 0,01 \\
= 3 \,000
\]

So this means that R3 000 of the first R3 300 repayment goes to the bank as an interest charge.

Therefore:

\[
\text{capital reduction} = \text{total repayment} - \text{interest amount} \\
= 3 \,300 - 3 \,000 \\
= 300
\]

Therefore, your home loan of R300 000 is reduced by only R300 (to R299 700) after the first month's repayment.

Let's now return to the subject of saving our money and watching it grow, rather than paying large amounts of interest in credit arrangements.

Regular savings

All of us have different reasons for saving. Some of us may be saving to put down a deposit on a home. Others may be saving for their education, or their children's education. We should all start saving for our retirement early, to ensure a reasonable income in old age.

The best way to save is to put aside some money on a regular basis. Even if it isn't a lot, regular monthly payments into a savings account or unit trust investment will grow faster and faster as time goes by. It is wise to put aside some money each month as an investment for the future.

Whatever our reasons for saving, we will want to know how fast our money is growing, and what the increased amount will be after a certain time period. Let's look at the way we can calculate this. Financial institutions calculate interest on savings using compound interest. Therefore, we will use compound interest in our calculations.

Remember that this is a positive factor when it comes to savings, as compound interest grows faster than simple interest!
The formula we use to calculate the future value \((F)\) of our savings is:

\[
F = s \times \frac{(1 + i)^n - 1}{i}
\]

where:

- \(s\) is the regular savings payment we make each period;
- \(i\) is the interest rate we earn on our savings; and
- \(n\) is the number of periods over which we save.

Once again, it is important to remember to keep the interest rate \((i)\) and the number of time periods \((n)\) consistent. The time period you use for \(n\) depends on how often the interest is compounded.

The following example shows you how to use this formula to calculate the future value of savings.

**EXAMPLE**

You plan to save R300 per month. Your bank offers a nominal interest rate on savings of 8% per annum (per year), compounded monthly. How much will your savings be worth after four years?

Because you are making your payments on a monthly basis, we must first convert the interest rate \((i)\) and the periods \((n)\) to monthly units.

A nominal rate of 8% per annum gives us \(\frac{8}{12} = 0.67\%\) per month.

A period of four years gives us \(4 \times 12\) months = 48 months.

We have:

\[
s = 300
\]

\[
i = 0.67\% \text{ or } 0.0067
\]

\[
n = 48
\]

By substituting these figures into the formula, we get:

\[
F = s \times \frac{(1 + i)^n - 1}{i}
\]

\[
= 300 \times \frac{(1 + 0.0067)^{48} - 1}{0.0067}
\]

\[
= 300 \times \frac{1.378 - 1}{0.0067}
\]

\[
= 300 \times \frac{1.378}{0.0067}
\]
Therefore, after saving R300 per month for four years, we end up with a healthy amount of R16 925,40 when the interest is compounded monthly.

To work out the interest on your savings, simply deduct the amount saved from the amount including interest:

\[ 16\,925,40 - (R300 \times 48) = 2\,525,40. \]

Therefore, you have earned R2 525,40 of interest over the four-year period.

In the last part of this lesson, we look at a particular type of savings plan, called a retirement annuity.

**Retirement annuities**

We should all be saving for our retirement. Although it may seem far away, it takes a long time to build up a savings or investment account that is big enough to support us in our retirement. You could live another 30 years after you've retired. You might want to do things you've never had the time to do when you were younger. You may also require special medical care. That is why you have to start making provision for your retirement when you are young.

The most popular form of investment used to make provision for retirement is a retirement annuity.

A retirement annuity is a regular stream of monthly income payments paid to a retired person. This stream of payments is ongoing and stops only when the retired person dies.

Retirement annuities are an effective means of providing an income during our later years in life. The size of the regular monthly income payments depends directly on the size of the contributions we can afford. Therefore, it is very important to remember that we need to save hard now to build up a lump sum large enough to buy a reasonable stream of monthly income payments.

**Summary**

In this lesson, we have shown you a few examples of the influence that interest can have on your finances. Whenever you borrow or invest money, you will either pay or receive interest.

Saving for something and earning interest, as opposed to buying it on credit and paying interest, can make a huge difference to the final amount we pay for an item that we want or need. It is up to you to decide whether you can wait or not! In this lesson, you have learnt how to calculate the real price of all the interest you end up paying.

We have also looked at providing for your retirement. The more you invest in your retirement now, the more money you will have to make life easier for you when you are not earning any more.
Self-assessment Questions 9

Test your knowledge of this lesson by completing the self-assessment questions below. When you answer the questions, don’t look at the suggested answers that we give straight away. Look at them only after you’ve written your answers down, and then compare your answers with our answers.

1. Suppose you want to buy furniture by means of a hire-purchase agreement. The cash price of the furniture is R10 000. The furniture store calculates instalments over 24 months, and charges 2% interest per month.
   
   (a) What is your monthly instalment?
   (b) What is the total amount that you will pay over the 24 months?
   (c) How much interest do you pay over the 24 months?

2. You want to start saving for a deposit on a house. You decide to invest R400 in a savings account every month. If the savings account earns 1% interest per month, compounded monthly, how much money will you have saved after three years?

3. Why do we need to save for retirement now to be able to invest in a retirement annuity on retirement?

4. A bank grants you a R500 000 loan to buy a house. The bank charges an effective interest rate of 0,8% per month on the loan. If you are planning to repay the loan over 20 years, what will your monthly repayment be?

Suggested answers to Self-assessment Questions 9

1. (a) You know the following information:

   \[ A = \text{R10 000} \]
   \[ i = 2\% \text{ per month} \]
   \[ n = 24 \text{ months} \]

   The formula for calculating the instalment (S) is:

   \[ S = \frac{A \times i \times (1 + i)^n}{(1 + i)^n - 1} \]

   \[ = \frac{10000 \times 0,02 \times (1,02)^{24}}{(1,02)^{24} - 1} \]

   \[ = \frac{10000 \times 0,02 \times 1,608}{1,608 - 1} \]

   \[ = \frac{321,687}{0,608} \]

   \[ = 529,09 \]

   Therefore, the monthly instalment is R529,09.
(b) The total you will pay over 24 months is:

$$529.09 \times 24 = 12\,698.16.$$ 

(c) The amount of interest that you pay is the difference between the total amount that you pay over the 24 months and the original cash price of the furniture. Therefore, the interest is:

$$12\,698.16 - 10\,000 = 2\,698.16$$

2. You know the following information:

$$S = R400$$

$$i = 1\% \text{ per month}$$

$$n = 3 \times 12 = 36 \text{ months}$$

The formula to calculate the future value ($F$) of your savings is:

$$F = S \times \frac{(1+i)^n - 1}{i}$$

$$= 400 \times \frac{(1.01)^{36} - 1}{0.01}$$

$$= 57\,240$$

Therefore, the value of your savings account is R57 240 after three years.

3. The retirement annuity we buy when we retire depends on the amount of money we have at that time. We need to save hard over a long time to build up a healthy lump sum by the time we retire. The larger the lump sum we can pay at retirement, the larger the regular income payments we will receive during retirement.

4. You know the following information:

$$A = R10\,000$$

$$i = 0.8\% \text{ per month}$$

$$n = 20 \times 12 \text{ months} = 240 \text{ months}$$

The formula to calculate the monthly repayment ($S$) is:

$$S = \frac{A \times i \times (1+i)^n}{(1+i)^n - 1}$$

$$= \frac{500\,000 \times 0.008 \times (1.008)^{240}}{(1.008)^{240} - 1}$$
= \frac{500000 \times 0.008 \times 6.769}{6.769 - 1} \\
= \frac{27076}{5.769} \\
= 4693.36

Therefore, your monthly home loan repayment is R4 693.36.

Check your competence

Now that you have worked through this lesson, please check that you can perform the tasks below:

- I can calculate the value of \( n \) in the formula \( A = P(1 \pm i)^n \).
- I can apply knowledge of geometric series to solving annuity, bond repayment and sinking fund problems.
- I can critically analyse investment and loan options, and make informed decisions as to the best option(s) (including, pyramid and micro-lenders’ schemes).
- I can solve non-routine, unseen problems.
CONCLUSION

Congratulations! You’ve successfully reached the end of this unit, and have learnt about how to use numbers, and the relationships between numbers, to solve problems.

In the first part of the study unit, we focused on numbers themselves. Here, you learnt about indices (exponents) and surds. These number forms allow you to express complicated expressions in a simpler way. This added simplicity, plus the helpful laws that govern indices and surds, enables you to work with these numbers more easily.

Logarithms are an extension of exponents. And, just like exponents, there are useful laws that explain exactly how they work.

The second part of the study unit focused on how to apply the theory on numbers and number relationships to real-life problems. So here, we investigated problems related to business transactions like bonds, hire purchase and annuities. You learnt about simple and compound interest, and nominal and effective interest rates.

Mathematics forms one of the foundations of our everyday lives. We hope that you are encouraged to read further on topics that interest you, and to continue to learn about how to use mathematics in your day-to-day life.
REFERENCES

Leaflets

SARS, *Information With Regard to Income Tax.*

Study guides


Web references


Please make the following changes to the above-mentioned Study Unit.

Refer to page 60.

Insert the following outcomes at the bottom of the Learning Outcomes box.

<table>
<thead>
<tr>
<th>LO</th>
<th>Assessment Standard</th>
</tr>
</thead>
<tbody>
<tr>
<td>LO 1</td>
<td><strong>AS 12.2.1(a)</strong> Demonstrate the ability to work with various types of functions and relations including the inverses listed in the following Assessment Standard.</td>
</tr>
<tr>
<td></td>
<td><strong>AS 12.2.1(b)</strong> Demonstrate knowledge of the formal definition of a function.</td>
</tr>
</tbody>
</table>
|     | **AS 12.2.2(a)** Investigate and generate graphs of the inverse functions, in particular the inverses of:  
  \[ y = ax + q \]
  \[ y = ax^2 \]
  \[ y = a^x, a > 0 \] |
|     | **AS 12.2.2(b)** Determine which inverses are functions and how the domain of the original function needs to be restricted so that the inverse is also a function. |
Refer to page 70.

Insert the following information before the Summary heading on page 70.

Inverse functions

Functions

*Function*: a set of ordered pairs in which no two ordered pairs have the same x – coordinate.

*Domain of a function*: the set of all x – coordinates.

*Range of a function*: the set of all y – coordinates.

*Relation*: any set of ordered pairs. When a relation is represented graphically, it is fairly easy to see if it is a function or not. If it is a function, there will be no vertical line that intersects the graph more than once. Thus there is only one x – value for each y – value on the graph.

Let’s look at some examples.

### Examples

State whether or not each of the following are functions:

1. \{ (0;1), (1;2), (2;3), (3;4), (4;5) \}
   Yes, it is a function, because each ordered pair has a different x – coordinate

2. \{ (0;1), (1;2), (1;3), (3;4), (4;5) \}
   No, there are two ordered pairs that have the same x – coordinates (1;2) and (1;3)

3. ![Graph of a function]
   Place your ruler vertically on the page, move it from left to right and see how many times the ruler intersects the graph.
   Yes, it is a function, because no vertical line will intersects the graph more than once. The reason then for being a function is: There is only one x – value of each y – value on the graph.
4. Place your ruler vertically on the page, move it from left to right and see how many times the ruler intersects the graph.

No, it is not a function, because there is at least one vertical line that intersects the graph more than once. The reason for not being a function: There are at least two points on the graph which have the same x-coordinate.

\[ f(x) = 4x + 2 \]

\[ \therefore f(3) = 4(3) + 2 = 14 \] Substitute the x with 3. (3;14) is an ordered pair on the function.

\[ f(-1) = 4(-1) + 2 = -2 \] Substitute the x with -1. (-1;-1) is an ordered pair on the function.

\[ f(0) = 4(0) + 2 = 2 \] Substitute the x with 0. (0;2) is an ordered pair on the function and it is also the y-intercept of the graph, because it is where x-coordinate is equal to 0.

\[ f(x) = 0 \] You can ask yourself the question: For which x-value(s) will the function be 0 or for which x-value(s) will the y-value be 0?

\[ 4x + 2 = 0 \] Make the function equal to 0.

\[ 4x = -2 \] Solve the equation.

\[ x = -\frac{1}{2} \] \((-\frac{1}{2};0)\) is an ordered pair on the function and it is the x-value as well, because it is where the y-coordinate is equal to 0.

A function can be written as \[ y = f(x) \], which is notation meaning \( y \) is a function of \( x \). This is called the function notation.

Let's look at some examples.
Inverses of functions

**Inverse of a function:** This is the graph which is the reflection of the original function about the line \( y = x \).

The notation of a function is \( f^{-1} \). That -1 is not an exponent. It simply indicates the inverse of the function \( f \) here.

In transformations, we see that the rule for the reflection about the line \( y = x \) is \((x; y) \rightarrow (y; x)\). It is basically swopping the \( x \)– coordinates with the \( y \)– coordinates and vice versa.

To find the inverse of a function, you will do the same, because the inverse is the reflection about the line \( y = x \).

**Steps to find the inverse:**

1. Replace \( f(x) \) with \( y \), if it is written in the function notation.
2. Swop the \( x \) and \( y \) in the equation.
3. Solve for \( y \).
4. Replace the \( y \) with \( f^{-1}(x) \).

Let's work through some examples.

**EXAMPLES**

Find the inverse of the following functions:

1. \( f(x) = 2x + 6 \)
   
   Replace \( f(x) \) with \( y \), if it is written in the function notation.
   
   \( y = 2x + 6 \)
   
   Swop the \( x \) and \( y \) in the equation.
   
   \( x = 2y + 6 \)
   
   Solve for \( y \).
   
   \( x = 2y + 6 \)
   
   \( x - 6 = 2y \)
   
   \( \frac{1}{2}x - 3 = y \)
   
   Replace the \( y \) with \( f^{-1}(x) \).
   
   \( f^{-1}(x) = \frac{1}{2}x - 3 \)

2. \( f(x) = 2x^2 \)
   
   Replace \( f(x) \) with \( y \), if it is written in the function notation.
   
   \( y = 2x^2 \)
   
   Swop the \( x \) and \( y \) in the equation.
\[ x = 2y^2 \]
Solve for \( y \).
\[ x = 2y^2 \]
\[ \frac{x}{2} = y^2 \]
\[ y = \pm \sqrt{\frac{x}{2}} \]
Replace the \( y \) with \( f^{-1}(x) \).
\[ f^{-1}(x) = \pm \sqrt{\frac{x}{2}} \]

3. \[ f(x) = 2^x \]
Replace \( f(x) \) with \( y \), if it is written in the function notation.
\[ y = 2^x \]
Swop the \( x \) and \( y \) in the equation.
\[ x = 2^y \]
Solve for \( y \).
\[ y = \log_2 x \quad \text{In your study guide on p 61 you will find the interchanging} \]
\[ \text{between exponential functions and logarithms.} \]
Replace the \( y \) with \( f^{-1}(x) \).
\[ f^{-1}(x) = \log_2 x \]

---

**Drawing the graphs of inverses**

**NB**
Remember that the inverse and the function is symmetrical about the line \( y = x \).

Here are some examples for you.

---

**EXAMPLES**

Draw the inverses of the following functions:

1. \[ y = 2x - 1 \]
First draw the function, if it is not already given:
\[ x - \text{intercept: Put } y = 0 \]
\[ y = 2x - 1 \quad \frac{1}{2} \]
\[ 1 = 2x \]
\[ x = \frac{1}{2} \quad \therefore \left( \frac{1}{2}, 0 \right) \]
\[ y - \text{intercept: Put } x = 0 \]
\[ y = 2(0) - 1 \]
\[ y = -1 \quad \therefore (0; -1) \]

Draw the line \( y = x \)
Swop the coordinates, plot it on the set of axes and draw the graph.

\((\frac{1}{2},0) \rightarrow (0,\frac{1}{2})\)

\((0,-1) \rightarrow (-1,0)\)

2. Given \(h(x) = 4^x\). Draw the graph \(h^{-1}(x)\) on the same set of axes.

If \(h(x) = 4^x\), then \(h^{-1}(x) = \log_4 x\) (see example 3 of the inverses of functions).
Draw the line $y = x$

Now use the points on the graph or use your own points to find the corresponding points on the inverse. If there are no points indicated, find your own points by making $x = 0$, 1 or -1 in the equation of the function. In this example we already have one point $(0;1)$. If you need another point, make say $x = -1$, then $y = \frac{1}{4}$, so another point is $(-1;\frac{1}{4})$.

Now swap the coordinates

$(0;1) \rightarrow (1;0)$

$(-1;\frac{1}{4}) \rightarrow (\frac{1}{4};-1)$
**Domain and range of the inverse**

Study the last example to find the domain and range of $h(x)$ and $h^{-1}(x)$.

<table>
<thead>
<tr>
<th></th>
<th>Domain</th>
<th>Range</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h(x)$</td>
<td>$x \in R$</td>
<td>$y &gt; 0, y \in R$</td>
</tr>
<tr>
<td>$h^{-1}(x)$</td>
<td>$x &gt; 0, x \in R$</td>
<td>$y \in R$</td>
</tr>
</tbody>
</table>

**The restrictions on the function to make the inverse a function**

The next graphs represent the following:

\[ f(x) = x^2 \]

and

\[ f^{-1}(x) = \pm \sqrt{x} \]

Although $f(x)$ is a function, its inverse is not a function. (A vertical line intersects the graph at more than one place at a time.)

To make the inverse a function, there must be a restriction in the function, for example, you can take $x \geq 0$, then the graphs will be as follows:
Now, the graphs of $f(x) = x^2$, $x \geq 0$ and its inverse are functions.

You may also use the restriction $x \leq 0$:

Now, the graphs of $f(x) = x^2$, $x \leq 0$ and its inverse are functions.
Refer to page 71.

Insert the following questions before the Suggested answers to Self-assessment Questions 4 heading.

6. Given: \( f(x) = 2^x \):

   (a) Determine an equation for \( f^{-1} \).
   (b) Sketch on the set of axes given the graphs of \( f \) and \( f^{-1} \), showing clearly ALL intercepts with the axes.
   (c) Write down the domain of \( f^{-1} \).
   (d) For which values of \( x \) will \( f(x)f^{-1}(x) \leq 0 \)?

7. Drawn below is the graph of \( g(x) = -2x^2, x \leq 0 \). One of the points on the graph is \((-1;2)\).

   ![Graph of g(x)]

   (a) Determine the equation of \( g^{-1}(x) \).
   (b) Write down the range of \( g^{-1}(x) \).
   (c) Draw the graph of \( g^{-1}(x) \) on the set of axes given as \( g(x) \). Indicate the coordinates of a point other than \((0;0)\).
8. This diagram represents the following graphs:

\[ f(x) = 2^x \quad \text{and} \quad g(x) = -x^2 + 2x + 1 \]

(a) Write down the equation of \( f^{-1}(x) \).

(b) Sketch the graph of \( f^{-1}(x) \) on the system of axes given. Indicate the \( x \)-intercept and the coordinate of one other point on your graph.

(c) How can the domain of \( g \) be restricted so that \( g^{-1} \) will be a function?

9. The following is the graph of \( g(x) = -\log_3 x \).

(a) Write down the domain of \( g(x) \).

(b) What will the domain of \( g^{-1}(x) \) be?

(c) Write down the asymptote of \( g^{-1}(x) \).
(d) Write down the equation of $g^{-1}(x)$.

(e) Sketch the graph of $g^{-1}(x)$ on the sketch page.

(f) Explain the transformation of graph $g^{-1}(x)$ to the graph of $h(x) = \log_3 x$.

**Answer sheet for Self-assessment Questions 6 to 9**

6. (a) ………………………………………………………………………………………………………

(b) ………………………………………………………………………………………………………

(c) ………………………………………………………………………………………………………

(d) ………………………………………………………………………………………………………

7. (a) ………………………………………………………………………………………………………

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(b) ………………………………………………………………………………………………………
8. (a) ...........................................................................................................

(b) ...........................................................................................................

(c) ..............................................................................................................

9. (a) ...........................................................................................................

(b) .............................................................................................................

(c) .............................................................................................................
Refer to page 74.

Insert the following answers before the Check your competence box.

6. (a) \( f^{-1}(x) = \log_2 x \)
(c) \( x > 0 \)

(d) \( 0 < x \leq 1 \) It is where the \( y \)-values of one graph is positive and the other is negative.

7. (a) \( y = -2x^2 \)
\( x = 2y^2 \)
\(-\frac{x}{2} = y^2 \)
\( y = \pm \sqrt{-\frac{x}{2}} \) Remember \( x < 0 \), so there will be a positive under the square root

but \( y \leq 0 \) The restriction of the function was \( x \leq 0 \)

\( \therefore y = \sqrt{-\frac{x}{2}} \)

(b) \( y \leq 0, y \in R \)

(c)
8. (a) \( f^{-1}(x) = \log_2 x \)

(b) 

(c) Restrict the domain of to \( x \geq 0 \) or \( x \leq 0 \)

9. (a) \( x > 0 \)

(b) \( y > 0 \)

(c) \( y = 0 \)

(d) \( g^{-1}(x) = 3^{-x} = \left(\frac{1}{3}\right)^x \)

(e) 

(f) It is a reflection of \( g(x) \) about the \( x \)-axis.
Refer to page 74.

Insert the following text at the end of the check your competence box.

- I can demonstrate the ability to work with various types of functions and relations including the inverses listed in the following Assessment Standard.
- I can demonstrate knowledge of the formal definition of a function.
- I can investigate and generate graphs of the inverse functions, in particular the inverses of:
  \[ y = ax + q \]
  \[ y = ax^2 \]
  \[ y = a^x, \ a > 0 \]
- I can determine which inverses are functions and how the domain of the original function needs to be restricted so that the inverse is also a function.